4] **VOTING SYSTEMS (Part Three):**  
**SOME SHORTCOMINGS OF k-ALTERNATIVE SYSTEMS; k>2**

4.1) Name at least five desirable properties that k-Alternative Voting Systems, $k > 2$, should satisfy.

4.2) State the **Condorcet Winner Criterion (CWC).** (Page 292)

4.3) Show that Plurality Voting does not satisfy the CWC.  
(Hint: Assume that it does and produce a counter example)  
Consider the voting system with preference ballot matrix:

<table>
<thead>
<tr>
<th>Rank</th>
<th>3</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>B</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>2nd</td>
<td>C</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>3rd</td>
<td>A</td>
<td>B</td>
<td>B</td>
</tr>
</tbody>
</table>

4.4) Show that the Borda Count Method does not satisfy the CWC.  
(Hint: Assume that it does and produce a counter example)  
Consider the voting system with preference ballot matrix:

<table>
<thead>
<tr>
<th>Rank</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>3rd</td>
<td>C</td>
<td>A</td>
</tr>
</tbody>
</table>

a) Find Condorcet’s Winner: **A**  
AvB: A wins 3-2; AvC: A wins 3-2; BvC: B wins 5-0  
b) Find Borda Count Winner: **B**  
A = 2(3) + 1(0) + 0 = 6; B = 2(2) + 1(3) + 0 = 7; C = 2(0) + 1(2) + 0 = 2  
Since the Borda Count Winner is not the same as the Condorcet’s Winner, then the Borda Count Method does not satisfy the CWC.

4.5) Explain why Sequential Pair-wise Voting satisfies the CWC.

4.6) Show that the Hare System does not satisfy the CWC.  
(Hint: Assume that it does and produce a counter example)  
Consider the voting system with preference ballot matrix:

<table>
<thead>
<tr>
<th>Rank</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>C</td>
<td>B</td>
<td>D</td>
<td>A</td>
</tr>
<tr>
<td>2nd</td>
<td>D</td>
<td>D</td>
<td>A</td>
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<td>C</td>
</tr>
<tr>
<td>4th</td>
<td>A</td>
<td>C</td>
<td>C</td>
<td>D</td>
</tr>
</tbody>
</table>

**SOLUTION:**  
a) Using the Condorcet’s Method:  
AvB: B wins 9-4; AvC: A wins 8-5;  
AvD: D wins 12-1; BvC: B wins 8-5;  
BvD: D wins 8-5; CvD: D wins 7-6  
**CONDORCET WINNER: D**
Now use the Hare System and show that the winner is not the same as the Condorcet winner, D.

ROUND 1: A=1; B=4; C=5; D=3. Eliminate candidate A.

New Matrix:

<table>
<thead>
<tr>
<th>Rank</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
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<td>B</td>
<td>D</td>
<td>B</td>
</tr>
<tr>
<td>2nd</td>
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<td>B</td>
<td>C</td>
</tr>
<tr>
<td>3rd</td>
<td>B</td>
<td>C</td>
<td>C</td>
<td>D</td>
</tr>
</tbody>
</table>

ROUND 2: B=5; C=5; D=3
Eliminate candidate D.

New Matrix:

<table>
<thead>
<tr>
<th>Rank</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>C</td>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>C</td>
<td>C</td>
<td>C</td>
</tr>
</tbody>
</table>

Hare System Winner: B

Since the Hare System Winner is not the same as the Condorcet’s Winner, then the Hare System does not satisfy the CWC.

4.7) Show that the Plurality Runoff Method does not satisfy the CWC.
(Hint: Assume that it does and produce a counter example)
Consider the voting system with preference ballot matrix:

<table>
<thead>
<tr>
<th>Rank</th>
<th>5</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>C</td>
<td>D</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>B</td>
<td>D</td>
<td>B</td>
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<tr>
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<td>A</td>
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<tr>
<td>4th</td>
<td>A</td>
<td>C</td>
<td>A</td>
<td>C</td>
</tr>
</tbody>
</table>

4.8) Explain why Coomb’s Method satisfies the CWC. (Bonus Question)

4.9) Show that Approval Voting does not satisfy the CWC.
(Hint: Assume that it does and produce a counter example)
Consider the voting system with preference ballot matrix:

<table>
<thead>
<tr>
<th>Rank</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
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<td>A</td>
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<tr>
<td>2nd</td>
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<tr>
<td>4th</td>
<td>A</td>
<td>D</td>
<td>C</td>
<td>B</td>
</tr>
</tbody>
</table>

Voters approve their top two ranked candidates.
4.10) State the **Pareto Condition**. (Page 297)


4.12) Explain why Plurality Voting satisfies the Pareto Condition.

4.13) Explain why the Condorcet’s Method satisfies the Pareto Condition (HW)

4.14) Explain why the Borda Count Method satisfies the Pareto Condition.

4.15) Show that Sequential Pairwise Voting does not satisfy the Pareto Condition (Hint: Assume that it does and produce a counter example)

Consider the voting system with the given preference ballot matrix.

<table>
<thead>
<tr>
<th>Rank</th>
<th>2</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>D</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>C</td>
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<tr>
<td>3rd</td>
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<td>B</td>
</tr>
<tr>
<td>4th</td>
<td>A</td>
<td>D</td>
<td>C</td>
</tr>
</tbody>
</table>

**Sequencial Pair-Wise Voting**

Agenda: D-B-A-C

4.16) Explain why the Hare System satisfies the Pareto Condition.

4.17) Explain why the Plurality Runoff Method satisfies the Pareto Condition.

JUSTIFICATION: If every voter prefers candidate X over candidate Y, then Y can never be first preference in any ballot and has zero first-preference votes. Since in the Plurality Runoff Method candidates go to Round 2 based on the highest and second highest number of first-preference votes, then Y will never be able to go to Round 2 and will be eliminated in Round 1. Therefore, Y will never be among the winners.

4.18) Explain why Coomb’s Method satisfies the Pareto Condition. (Bonus Question)

4.19) If every voter approves their top three ranked candidates, use the given matrix to show that Approval Voting does not satisfy the Pareto Condition.

<table>
<thead>
<tr>
<th>Rank</th>
<th>2</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
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</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>3rd</td>
<td>C</td>
<td>B</td>
<td>E</td>
</tr>
<tr>
<td>4th</td>
<td>D</td>
<td>E</td>
<td>C</td>
</tr>
<tr>
<td>5th</td>
<td>E</td>
<td>C</td>
<td>D</td>
</tr>
</tbody>
</table>

JUSTIFICATION:

- A has 5 approvals; B has 5 approvals;
- C has 2 approvals; D has 2 approvals;
- E has 1 approval.

There is no single winner. Candidates A,B tied on first place as winners. However, every voter prefers A over B, so B should not be among the winners.
4.20) State the Independence of Irrelevant Alternatives, IIA. (Page 295)

4.21) Explain why Plurality Voting does not satisfy the IIA.
(Hint: Assume that it does and produce a counter example)
Consider the voting system with preference ballot matrix:

<table>
<thead>
<tr>
<th>Rank</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>C</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>3rd</td>
<td>C</td>
<td>B</td>
<td>A</td>
<td>C</td>
</tr>
</tbody>
</table>

4.22) Explain why the Condorcet’s Method satisfies the IIA.

4.23) Explain why the Borda Count Method does not satisfy the IIA.
(Hint: Assume that it does and produce a counter example)
Consider the voting system with preference ballot matrix:

<table>
<thead>
<tr>
<th>Rank</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>C</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>2nd</td>
<td>C</td>
<td>A</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>3rd</td>
<td>B</td>
<td>B</td>
<td>A</td>
<td>C</td>
</tr>
</tbody>
</table>

4.24) Use the given ballot matrix to show that Sequential Pair-wise Voting does not satisfy IIA. Use agenda: D-B-A-C. (HOMEWORK)

<table>
<thead>
<tr>
<th>Rank</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>D</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
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<td>1</td>
</tr>
<tr>
<td>3rd</td>
<td>C</td>
<td>A</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4th</td>
<td>A</td>
<td>D</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

INITIAL ELECTION:

NEW ELECTION: Switch ranks of B, D (shown in red)

NEW WINNER:
4.25) Explain why the Hare System does not satisfy the IIA.
(Hint: Assume that it does and produce a counter example)
Consider the voting system with preference ballot matrix:

<table>
<thead>
<tr>
<th>Ranks</th>
<th>4</th>
<th>3</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>C</td>
<td>B</td>
<td>D</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>D</td>
<td>C</td>
<td>B</td>
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<tr>
<td>3rd</td>
<td>C</td>
<td>A</td>
<td>D</td>
<td>A</td>
</tr>
<tr>
<td>4th</td>
<td>D</td>
<td>B</td>
<td>A</td>
<td>C</td>
</tr>
</tbody>
</table>

4.26) Use the given ballot matrix to show that Plurality Runoff does not satisfy IIA

<table>
<thead>
<tr>
<th>Rank</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>C</td>
<td>D</td>
<td>B</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>D</td>
<td>C</td>
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<tr>
<td>3rd</td>
<td>C</td>
<td>B</td>
<td>B</td>
<td>D</td>
</tr>
<tr>
<td>4th</td>
<td>D</td>
<td>A</td>
<td>A</td>
<td>C</td>
</tr>
</tbody>
</table>

**NEW MATRIX**

<table>
<thead>
<tr>
<th>Rank</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>C</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>2nd</td>
<td>C</td>
<td>A</td>
<td>A</td>
<td>C</td>
</tr>
</tbody>
</table>

**NEW ELECTION**

In the original matrix, switch the ranks of B-C to obtain:

<table>
<thead>
<tr>
<th>Rank</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>B</td>
<td>D</td>
<td>B</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>D</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>3rd</td>
<td>C</td>
<td>C</td>
<td>B</td>
<td>D</td>
</tr>
<tr>
<td>4th</td>
<td>D</td>
<td>A</td>
<td>A</td>
<td>C</td>
</tr>
</tbody>
</table>

**NEW MATRIX**

<table>
<thead>
<tr>
<th>Rank</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
</tbody>
</table>

By Plurality, the new winner is B. So, B went from loser to winner status without switching ranks with original winner A.

4.27) (BONUS) Explain why Coomb’s Method does not satisfy the IIA.
(Hint: Assume that it does and produce a counter example)
Consider the voting system with preference ballot matrix:

<table>
<thead>
<tr>
<th>Rank</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>C</td>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td>2nd</td>
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<tr>
<td>3rd</td>
<td>C</td>
<td>B</td>
<td>A</td>
<td>A</td>
</tr>
</tbody>
</table>
4.28) If every voter approves the top two preferences, use the given ballot matrix to show that Approval Voting does not satisfy the IIA. (HOMEWORK)

<table>
<thead>
<tr>
<th>Rank</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
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<tr>
<td>2nd</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>3rd</td>
<td>C</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
</tbody>
</table>

ORIGINAL ELECTION

4.29) State Monotonicity for three-or-more candidate systems. (Page 299)

4.30) Explain why Plurality Voting satisfies Monotonicity.

4.31) Explain why the Condorcet’s Method satisfies Monotonicity (HOMEWORK)

4.32) Explain why the Borda Count Method satisfies Monotonicity.

4.33) Explain why Sequential Pairwise Voting satisfies Monotonicity.

JUSTIFICATION: Suppose candidate X is the winner of an election that used Sequential Pair-wise Voting. If a new election is held and at least one voter moves X to a higher rank, then X will defeat by a larger margin all candidates that it originally defeated in the first election. The schedule one-on-one contests that do not involve X on the agenda will have the same outcomes because the ranks of the rest of the pairs of candidates will not be reversed. Therefore, X will still be the winner of the election.

4.34) Explain why the Hare System does not satisfy Monotonicity.

(Hint: Assume that it does and produce a counter example)

Consider the voting system with preference ballot matrix:

<table>
<thead>
<tr>
<th>Rank</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
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<tbody>
<tr>
<td>1st</td>
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<td>C</td>
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<td>3rd</td>
<td>C</td>
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<td>C</td>
</tr>
</tbody>
</table>

HOMEWORK

4.35) Use the given matrix to show that Plurality Runoff does not satisfy Monotonicity.

<table>
<thead>
<tr>
<th>Rank</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
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<td>5th</td>
<td>E</td>
<td>E</td>
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<td>E</td>
<td>A</td>
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</tbody>
</table>
NEW MATRIX

<table>
<thead>
<tr>
<th>Rank</th>
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<th>3</th>
<th>3</th>
<th>2</th>
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</tr>
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<td>1^{st}</td>
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<td>2^{nd}</td>
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<td>3^{rd}</td>
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</tr>
</tbody>
</table>

Round 2:

NEW ELECTION: Move the winner ___ to a higher rank to obtain:

<table>
<thead>
<tr>
<th>Rank</th>
<th>4</th>
<th>3</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
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<tbody>
<tr>
<td>1^{st}</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>B</td>
<td>E</td>
</tr>
<tr>
<td>2^{nd}</td>
<td>B</td>
<td>A</td>
<td>A</td>
<td>D</td>
<td>D</td>
</tr>
<tr>
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4.36) (BONUS) Explain why Coomb’s Method satisfies Monotonicity.
4.37) Explain why Approval Voting satisfies Monotonicity. (HOMEWORK)
4.38) Describe Manipulability and Insincere Ballots. (Page 292)
4.39) Majority Rule is a non-manipulable two-alternative voting system. (Just know it)
4.40) Minority Rule is a manipulable two-alternative voting system. (Just know it)
4.41) Imposed Rule is a non-manipulable two-alternative voting system. (Just know it)
4.42) Dictatorships are a non-manipulable. (Just know it)
4.43) The two-alternative voting system in which the winner is the candidate with an odd number of first-preference votes is manipulable. (Just know it)
4.44) The two-alternative voting system in which the winner is the candidate with an even number of first-preference votes is manipulable. (Just know it)
4.45) Plurality Voting is manipulable. (Just know it)
4.46) A Condorcet Paradox is manipulable. (Just know it)
4.47) If there is a Condorcet winner, the Condorcet’s Method is non-manipulable. (Just know it)
4.48) The Borda Count Method with exactly three candidates is non-manipulable. (Just know it)
4.49) The Borda Count Method with four or more candidates is manipulable. (Just know it)
4.50) Sequential Pairwise Voting is manipulable. (Just know it)
4.51) The Hare System is manipulable. (Just know it)
4.52) The Plurality Runoff Method is manipulable. (Just know it)
4.53) Coomb’s Method is manipulable. (Just know it)
4.54) Approval Voting is manipulable. (Just know it)
4.55) Describe other ways to manipulate voting systems.
4.56) State the strong version of Arrow’s Impossibility Theorem.
4.57) State the weak version of Arrow’s Impossibility Theorem. Why is this a “weak version”?
4.59) Summary matrix of systems and desirable properties

<table>
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<th>SYSTEM</th>
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