3.1) Describe Weighted Voting Systems, WVS.
(Measures, Motions, Resolutions, Propositions)
(Players, Voter’s Weight, Quota)

3.2) Mathematical Notation of WVS.
Examples:
3.2.1) Describe the WVS: [6: 4, 3, 1]
3.2.2) A tiny corporation consists of five stockholders: Nancy (holds 20 votes), Sy (15 votes), Amal (10 votes), Tyler (8 votes), and Angela (3 votes). The five members decide to form a Board of Directors to oversee the company and agree that it takes 30 or more votes to pass a motion. Represent this weighted voting system mathematically.
3.2.3) Describe the WVS: [37: 20, 18, 14, 12, 8]
SOLUTION
[37: 20, 18, 14, 12, 8] is a weighted voting system with quota 37 and five voters having voting weights 8, 12, 14, 18, and 20, respectively. A resolution passes if the total number of votes of all voters who favor the resolution is 37 or more.

3.3) Restrictions on the Quota q.
Examples:
3.3.1) What is wrong with the WVS [2: 1, 3, 5]?
3.3.2) What is wrong with the WVS [10: 1, 3, 5]?
3.3.3) List all possible values of q in the WVS: [q : 6, 4, 3, 1]
3.3.4) What are the values that q cannot have in the WVS [q : 8, 6, 5, 3, 2, 1]?
3.3.5) The United States Electoral College is a good example of a WVS. In presidential elections, voters cast their votes for the electors that will represent them in the Electoral College, which consists of 51 member participants: the 50 States in the Union and the District of Columbia (23rd Amendment). The weights range from 1 to 55 electoral votes, depending on the population size of the State’s Congressional Delegation. The overall sum of the electoral votes allotted to the 51 participants is 538. The presidential candidate who first reaches the quota q is declared the winner of the presidential election (the new president of the United States). Find the minimum value of q. **ANSWER:** By simple majority, $q = 270$ electoral votes.

3.4) Describe Dictators in a WVS.
3.4.1) Consider the WVS: $[8: 8, 5, 2]$
3.4.2) Consider the WVS: $[70: 65, 75]$
3.4.3) Consider the WVS: $[17: 6, 10, 15, 16]$

3.5) Describe Dummy Voters in a WVS.
3.5.1) Consider the WVS: $[6: 7, 3, 1]$
3.5.2) Consider the WVS: $[8: 5, 4, 2]$
3.5.3) Consider the WVS: $[10: 5, 5, 3, 1]$
3.5.4) The WVS $[7: 5, 5, 2]$ does not have dummy voters. Why?

3.6) Describe Veto Power in a WVS.
3.6.1) Consider the WVS: $[7: 6, 3, 2, 1]$
3.6.2) Consider the WVS: $[8: 9, 4, 2]$
3.6.3) Consider the WVS: $[5: 4, 3, 1]$
3.6.4) In a WVS, two or more voters may have veto power. Consider the system $[6: 4, 3, 1]$. 


3.6.5) In a jury trial, the 12 members of the jury must reach a unanimous decision to find a person guilty. So, each juror has veto power.

3.6.6) It is also possible that no voter has veto power. Consider the system [9: 5, 5, 4]

3.6.7) Explain the (main) difference between a dictator and a voter with veto power (who is not a dictator). No examples


3.7.1) Dictators
3.7.2) Jurors of a court trial
3.7.3) Underage children
3.7.4) Students in a class
3.7.5) Stockholders of a Corporation
3.7.6) A player allowed more votes than another player does not necessarily have more voting power. Consider the systems [16: 8, 8, 7] and [20: 10, 9, 1]. Explain.

3.8) Measures of Voting Power.

3.9) Describe the Shapley-Shubik Power Index (SSPI)

3.10) Define Sequential Coalitions. Examples.

3.10.1) Find the number of sequential coalitions in the WVS: [8: 6, 4, 2]. List all possible sequential coalitions.

3.10.2) Compute the number of sequential coalitions in the WVS: [7: 5, 3, 2, 1]. List all the sequential coalitions.


3.11.1) List the sequential coalitions in which the 8-vote player is pivotal in the WVS: [8: 6, 4, 2].

3.11.2) List the sequential coalitions in which the 2-vote player is pivotal in the WVS: [7: 5, 3, 2, 1]

3.12) Computing the Shapley-Shubik Power Index

3.12.1) In each part, compute the SSPI for the given WVS:

a) [8: 6, 4, 2]   b) [15: 10, 8, 7]   c) [10: 7, 4, 2]
### a) [8:6,4,2]

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<tr>
<th>Sequential Coalitions</th>
<th>Cumulative Weights</th>
<th>Pivotal Voters</th>
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\[ SSPI(A) = SSPI(B) = SSPI(C) \]

**System SSPI =**

### b) [15:10,8,7]

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<th>Pivotal Voters</th>
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\[ SSPI(A) = SSPI(B) = SSPI(C) \]

**System SSPI =**

### SOLUTION TO PART c)

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<tr>
<td>A C B</td>
<td>7 9 13</td>
<td>B</td>
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<tr>
<td>B A C</td>
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<tr>
<td>B C A</td>
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<td>C A B</td>
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<tr>
<td>C B A</td>
<td>2 6 13</td>
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\[ SSPI(A) = \frac{3}{6} = \frac{1}{2}; \quad SSPI(B) = \frac{3}{6} = \frac{1}{2}; \quad SSPI(C) = \frac{0}{6} = 0; \quad System SSPI = \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \]
3.12.2) Compute the SSPI for $[12: 8, 6, 5, 3]$

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<th>Pivotal Voters</th>
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<td>8 14 17 22</td>
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<td>D C B A</td>
<td>3 8 14 22</td>
<td>B</td>
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SSPI(A) = \frac{10}{24} = \frac{5}{12}; \quad SSPI(B) = \frac{6}{24} = \frac{1}{4}; \quad SSPI(C) = \frac{6}{24} = \frac{1}{4}; \quad SSPI(D) = \frac{2}{24} = \frac{1}{12}

Therefore, System SSPI = \left( \frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12} \right)
3.12.3) The Paradox of New Members. Example.

3.12.4) Verify that the SSPI for \([15: 8, 6, 3]\) is

\[
\text{System SSPI} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)
\]

3.12.5) Verify the SSPI for \([15: 8, 6, 3, 1]\)

\[
\text{System SSPI} = \left(\frac{5}{12}, \frac{5}{12}, \frac{1}{12}, \frac{1}{12}\right)
\]

COMMENT


1) Every SSPI value is a positive real number in the closed interval \([0, 1]\). If SSPI(X) = 0, then voter X is a dummy voter. If SSPI(X) = 1, then voter X is a dictator. The SSPI values of a WVS represent a probability distribution that describes the distribution of the voting power among the voters in the WVS.

2) The sum of SSPI values of all voters in a WVS is always 1.

3) Voters with equal weights have equal SSPI values.

3.14) If a WVS has SSPI=(1/4, 1/5, 3/10, \(x\)), find the value of \(x\).

3.15) Find the SSPI for the WVS: \([8: 3, 1, 1, 1, 1, 1]\)

3.16) Find the SSPI for the WVS: \([12: 5, 2, 2, 2, 2, 2, 2, 2]\)

3.17) Two Co-Chairs:

a) Compute the System SSPI: \([8: 3, 3, 1, 1, 1, 1, 1]\)

b) Compute the System SSPI: \([10: 3, 3, 1, 1, 1, 1, 1, 1, 1, 1]\)

c) Compute the System SSPI: \([7: 3, 3, 1, 1, 1, 1, 1]\)

SOLUTION

There are 7! = 5,040 sequential coalitions (permutations).

Let Y represent one of the 1-vote players. For \(q = 7\), Y is pivotal as follows:

1) In every permutation of the form (3,3,Y,1,1,1,1).

There are 2! = 2 ways of placing the two co-chairs 4! = 24 ways of placing the four voters on the right hand side of Y, (Y is fixed). So far, there are (2!)(4!) = 48 permutations in which Y is pivotal.
2) In permutations of the form (3,1,1,1,Y,3,1).
   Now, there are 4! = 24 ways of placing the four voters on the left and 2
   ways to place the two voters on the right hand side of Y. The chairman on
   the left is one of the two and the three regular voters are any three of four.
   There are (4!)(2!)(2C1)(4C3) = 384 additional permutations in which Y is
   pivotal. So, Y is pivotal in 432 permutations and SSPI (Y) = \[ \frac{432}{5040} = \frac{3}{35} \]
   By property 3, each of the five voters with weight 1 has SSPI = \[ \frac{3}{35} \]
   Since the sum of the SSPI’s is 1, then each chairman has SSPI = \[ \frac{2}{7} \]

3) Therefore, System SSPI = \[ \left( \frac{2}{7}, \frac{2}{7}, \frac{3}{35}, \frac{3}{35}, \frac{3}{35}, \frac{3}{35}, \frac{3}{35} \right) \]

3.18) Describe the Banzhaf Power Index, (BPI)
3.19) Define: Coalition, Sequential Coalition, Winning Coalition,
       Blocking Coalition, Losing Coalition, Threshold Value.
3.20) For the WVS: [40: 30, 25, 15], compute the total number of
       coalitions and list all the winning, blocking, and the losing
       coalitions. Use set braces {} for each coalition.
3.21) For the WVS: [8: 4, 3, 2, 1], compute the total number of
       coalitions and list all the winning, blocking, and the losing
       coalitions. Use set braces {} for each coalition.
       Consider the WVS: [10: 7, 5, 3, 2]. Use A for the 7-vote player, B
       is the 5-vote player, C is the 3-vote player, D is the 2-vote player:
       a) find the critical voters in the winning coalition {A,B,C}
       b) find the critical voters in the blocking coalition {B,C,D}
3.23) Define Minimal Coalitions (MWC’s, MBC’s). Examples.
       Consider the WVS: [10: 7, 5, 3, 2]. Use A for the 7-vote player, B
       is the 5-vote player, C is the 3-vote player, D is the 2-vote player:
       a) Is {A,B,C} a MWC? b) Is {B,C,D} a MWC?
       c) Is {A,B} a MBC?
   a) Extra Votes of a Winning Coalition.
   b) Extra Votes of a Blocking Coalition.
   c) Critical Voters and Extra Votes.

3.26) Finding the BPI of a WVS:
   3.26.1) Find the BPI of the WVS: [14: 8, 6, 5]. Let \( A \) represent
           the 8-vote player, \( B \) is the 6-vote player, \( C \) is the 5-vote player. Use set braces correctly. Points will be deducted for not using correct set brace notation.
           SOLUTION
           a) COALITIONS:
           b) WEIGHTS:
           c) Threshold Value of Winning Coalitions, WC:
           d) Threshold Value of Blocking Coalitions, BC:

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BPI(A) =; BPI(B) =; BPI(C) =.
System BPI = Normalized Percent BPI =

3.26.2) Find the BPI of the WVS: [15: 8, 6, 5, 3]. Let \( A \) represent the 8-vote player, \( B \) is the 6-vote player, \( C \) is the 5-vote player, \( D \) is the 3-vote player. Use set braces correctly. Points will be deducted for not using correct set brace notation.
           SOLUTION…
3.27) Properties of BPI Values.

3.28) Larger WVS (one Chair): \textbf{(BONUS)} Compute the System BPI
\textbf{a)} \([6: 4, 1, 1, 1, 1, 1]\); \textbf{b)} \([7: 3, 1, 1, 1, 1, 1, 1]\); \textbf{c)} \([5: 3, 1, 1, 1, 1, 1, 1]\)

\textbf{SOLUTION TO PART c)}
Since \(q=5\), every winning coalition must have total weight 5 or more.
\textbf{a)} The 3-vote player is critical when contained in winning coalitions with
weights 5, 6, or 7, with the forms: \{3,1,1\}, \{3,1,1,1\}, and \{3,1,1,1,1\}.
Count: \(5C2 + 5C3 + 5C4 = 50\) winning coalitions. By the WBD, we double this
number and therefore, the 3-vote player is critical in a total of 100 coalitions
(winning plus blocking coalitions).
\textbf{b)} Let X represent one of the 1-vote players. Then X is critical when contained
in winning coalitions with weight 5 only, which have the form \{3,X,1\} and
\{X,1,1,1,1\}. The 1’s are from the remaining five 1-vote players. So, we
count: \(5C1 + 5C4 = 10\) winning coalitions. We double this number and
therefore, every 1-vote player is critical in a total of 20 coalitions (winning
plus blocking coalitions).
\textbf{c)} Now, the System BPI = (100, 20, 20, 20, 20, 20, 20).

3.29) \textbf{(BONUS)} \textbf{MORE SOLVED EXAMPLES}
\textbf{3.29.1)} Compute the System BPI: \([20: 6, 3, 3, 3, 3, 3, 3, 3, 3, 3]\)
\textbf{SOLUTION}
Since \(q=20\), every winning coalition must have total weight 20 or more.
\textbf{a)} The 6-vote player is critical when contained in winning coalitions with total
weights 20, 21, 22, 23, 24 or 25. The possible forms are only: \{6,3,3,3,3,3\}
and \{6,3,3,3,3,3,3\}. Now count: \(9C5 + 9C6 = 210\) winning coalitions. By the
WBD, we double this number, so the 6-vote player is critical in 420 coalitions.
\textbf{b)} Let X represent one of the 3-vote players. Then X is critical when contained
in winning coalitions with weights 20, 21, or 22.
Possible forms: \{6,X,3,3,3,3\} and \{X,3,3,3,3,3,3\}. The 3’s come from the
remaining eight 3-vote players. Count: \(8C4 + 8C6 = 98\) winning coalitions. By the
WBD and by 3), we double this number and therefore, every 3-vote
player is critical in a total of 196 coalitions.
\textbf{c)} System’s BPI = (420, 196, 196, 196, 196, 196, 196, 196, 196).

Normalized BPI \[\left(\frac{5}{26}, \frac{7}{28}, \frac{7}{28}, \frac{7}{28}, \frac{7}{28}, \frac{7}{28}, \frac{7}{28}, \frac{7}{28}, \frac{7}{28}\right)\]
3.29.1) COMMITTEES WITH TWO CO-CHAIRS

Compute the System BPI: \([7: 3, 3, 1, 1, 1, 1, 1, 1]\)

SOLUTION

Since \(q=7\), every winning coalition must have weight 7 or more.

a) Let \(Y\) represent one of the 3-vote players. Then \(Y\) is critical when contained in winning coalitions with weights 7, 8, or 9. The possible forms are:

\[
\{Y, 3, 1\}; \{Y, 3, 1, 1\}; \{Y, 3, 1, 1, 1\}; \{Y, 1, 1, 1, 1, 1\}; \{Y, 1, 1, 1, 1, 1, 1\}
\]

Count: \(6C_1 + 6C_2 + 6C_3 + 6C_4 + 6C_5 + 6C_6 = 63\) winning coalitions. By the WBD, we double this number, so each of the 3-vote players is critical in 126 coalitions.

b) Let \(X\) represent one of the 1-vote players. Then \(X\) is critical if contained in winning coalitions with weight 7, with the forms \(\{3, 3, X\}\) and \(\{X, 3, 1, 1, 1\}\)

Count: \(1 + (2C_1)(5C_3) = 21\) winning coalitions. We double this number and therefore, every 1-vote player is critical in a total of 42 coalitions.

c) System’s BPI = (126, 126, 42, 42, 42, 42, 42).

Normalized BPI = \(\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)\)

3.30) Consider the WVS: \([14: 7, 6, 4, 3, 1, 1]\)

(a) Find the number of possible subsets of voters
(b) Find the number of possible sequential coalitions.
(c) Find the threshold value of a winning coalition.
(d) Find the threshold value of a blocking coalition.

3.31) Classification of WVS with Two, Three, Four Members.

3.31.1) Two-Member WVS: Dictatorship; Consensus.

a) DICTATORSHIP. Find the MC’s for: \([2: 2, 1]\). Use \(A\) for the 5-vote player, \(B\) for the 3-vote player.

MWC’s: only the dictator’s coalition, \{\(A\)\}

MBC’s: only the dictator’s coalition, \{\(A\)\}

b) CONSENSUS (UNANIMITY). Find the MC’s for \([2: 1, 1]\). Use \(A\) for one 1-vote player; \(B\) for the other.

MWC’s: \{\(A, B\)\}; MBC’s: \{\(A\)\}; \{\(B\)\}; \{\(A, B\)\}
3.31.2) Three-Member WVS: Dictatorship; Consensus; Clique; Chair Veto; Majority.

a) **DICTATORSHIP.** Find the MC’s for: [3: 3, 1, 1]. Use A for the 3-vote player, B for one 1-vote player; C for the other. MWC’s: only {A}; MBC’s: only {A}.

b) **CONSENSUS (UNANIMITY):** Find the MC’s for: [3: 1, 1, 1]. Use A for one 1-vote player, B for the second, C for the third. MWC’s: only {A,B,C}; MBC’s: {A}, {B}, {C}.

c) **MAJORITY.** Find the MC’s for: [2: 1, 1, 1]. Use A for one 1-vote player, B for the second, C for the third. MWC’s: {A,B}, {A,C}, {B,C}; MBC’s: {A,B}, {A,C}, {B,C}.

d) **CLIQUE.** Find the MC’s for: [4: 2, 2, 1]. Use A for one 2-vote player, B for the second, C for the 1-vote player. MWC’s: only {A,B}; MBC’s: {A}, {B}.

e) **CHAIR VETO.** Find the MC’s for: [3: 2, 1, 1]. Use A for the 2-vote player, B for one 1-vote player, C for the second. MWC’s: {A,B}, {A,C}; MBC’s: {A}, {B,C}.

3.32) Classify the given WVS’s as dictatorship, unanimity, clique, majority, or chair-veto: a) [6: 4, 3, 1]; b) [5: 4, 3, 2]; c) [5: 4, 3, 1]; d) [9: 5, 3, 2]; e) [5: 5, 3, 1]

3.33) Define Equivalent Systems.

3.33.1) Show that [3: 2, 1, 1] and [11: 7, 5, 4] are equivalent WVS’s

3.33.2) Show that [6: 4, 3, 2, 1] and [20: 15, 10, 5, 4] are not equivalent WVS’s

3.34) In each case, explain why the WVS’s are not the same, but are equivalent: a) [7:3, 9] and [2:2, 1]; b) [6: 5, 3, 2] and [100: 20, 75, 90]; c) [8:3, 2, 2, 2] and [50: 35, 6, 5, 4]
3.35) SUMMARY OF TWO-VOTER WVS’s:
Every two-voter WVS is equivalent to either a two-voter dictatorship or to a two-voter unanimity (consensus).

<table>
<thead>
<tr>
<th>System</th>
<th>Simplest Form</th>
<th>MWC</th>
<th>BPI</th>
<th>SSPI</th>
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<td>(1, 0)</td>
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<td>Consensus</td>
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<td>(2, 2)</td>
<td>(½, ½)</td>
</tr>
</tbody>
</table>

a) Equivalent two-member dictatorships:

[3: 3, 2]; [5: 6, 3]; [15: 20, 4]; [100: 80, 100]; [50: 35, 60] are all different but equivalent to [2: 2, 1].

b) Equivalent two-member consensus (unanimities):

[10: 5,5]; [60: 59,2]; [3:2,2]; [5:1,4]; [15:8,7]; [100:1,99] are all different but equivalent to [2: 1, 1].

3.36) SUMMARY OF THREE-VOTER WVS’s:
Every three-voter WVS is equivalent to either a three-voter dictatorship or to a three-voter unanimity (consensus) or to a three-voter clique or to a three-voter majority or to a three-voter chair veto WVS.

<table>
<thead>
<tr>
<th>System</th>
<th>Simplest Form</th>
<th>Minimal Winning Coalitions</th>
<th>BPI</th>
<th>SSPI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dictatorship</td>
<td>[3: 3, 1, 1]</td>
<td>{A}</td>
<td>(8,0,0)</td>
<td>(1, 0, 0)</td>
</tr>
<tr>
<td>Consensus</td>
<td>[3: 1, 1, 1]</td>
<td>{A, B, C}</td>
<td>(2,2,2)</td>
<td>(½, ½, ½)</td>
</tr>
<tr>
<td>Chair Veto</td>
<td>[3: 2, 1, 1]</td>
<td>{A, B}, {A, C}</td>
<td>(6,2,2)</td>
<td>(2/3, 1/6, 1/6)</td>
</tr>
<tr>
<td>Clique</td>
<td>[4: 2, 2, 1]</td>
<td>{A, B}</td>
<td>(4,4,0)</td>
<td>(½, ½, 0)</td>
</tr>
<tr>
<td>Majority</td>
<td>[2: 1, 1, 1]</td>
<td>{A,B},{A,C},{B,C}</td>
<td>(4,4,4)</td>
<td>(1/3, 1/3, 1/3)</td>
</tr>
</tbody>
</table>

a) Equivalent three-member dictatorships:

[10: 12, 5, 2]; [5: 5, 3, 1]; [13: 14, 7, 2]; [50: 60, 25, 10]; [10: 3, 5, 10]; [9: 10, 4, 2] are all different WVS’s but equivalent to [3:3,1,1].
b) Equivalent three-member **consensus** (**unanimities**):
[14: 7, 6, 2]; [5: 2, 2, 2]; [10: 5, 3, 3]; [40: 20, 19, 1]; [4: 2, 1, 1];
[999: 250, 350, 400]; are all different WVS’s but equivalent to [3:1,1,1].

c) Equivalent three-member **chair veto’s**:
[5: 3, 2, 2]; [12: 8, 6, 5]; [26: 23, 4, 3]; [50: 35, 25, 20];
[60:40,35,20] are all different but equivalent to [3:2,1,1].

d) Equivalent three-member **cliques**:
[5: 3, 2, 1]; [14: 9, 6, 3]; [45: 30, 20, 10]; [125: 75, 60, 45];
[6: 3, 3, 2] [50:30,20,10] are all different but equivalent to [4:2,2,1].

e) Equivalent three-member **majorities**:
[5: 3, 3, 2]; [9: 5, 5, 5]; [60: 45, 35, 25]; [200: 150, 120, 100];
[3: 2, 2, 2] [18:12,10,8] are all different but equivalent to [2:1,1,1].

**3.37) FOUR-MEMBER EQUIVALENT SYSTEMS**

It may be possible to summarize four-member weighted voting systems into a table similar to what we did for two- and three-member systems. However, as the number of members increases, the number of different types of equivalent weighted voting systems increases. For the case of four members, there seem to be 14 different types of WVS. We may (later) attempt to list as many as possible.

**3.37.1) Show that the systems are equivalent and compute the BPI and the SSPI for each:**
[11: 7, 5, 2, 2] and [5: 3, 2, 1, 1].

**SOLUTION**

Verify that the corresponding winning coalitions are:
{A,B}, {A,B,C}, {A,B,D}, {A,C,D}, {A,B,C,D}
Verify also: BPI=(10, 6, 2, 2) and SSPI=\[\begin{pmatrix}
\frac{7}{12}, & \frac{1}{4}, & \frac{1}{12}, & \frac{1}{12}
\end{pmatrix}\]
3.37.2) Show that the systems are equivalent and compute the BPI and the SSPI for each:
[15: 12, 9, 4, 2] and [6: 4, 3, 2, 1].

SOLUTION

Verify that the corresponding winning coalitions are:
\{A,B\}, \{A,C\}, \{A,B,C\}, \{A,B,D\},
\{A,C,D\}, \{B,C,D\}, \{A,B,C,D\}

Verify also: BPI=(10, 6, 6, 2) and SSPI=\left(\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}\right)

3.38) Paradox of New Members. Examples.

In item 3.12.3, we described the Paradox of New Members. Examples 3.12.4, 3.12.5 illustrate an instance in which the paradox occurs when calculating the SSPI of a WVS. The paradox occurs also when calculating the BPI. Examine the two examples below.

3.38.1) Verify the BPI for \[15: 8, 6, 3\] is

BPI = \(2, 2, 2\) = \(33 \frac{1}{3}\%\), \(33 \frac{1}{3}\%\), \(33 \frac{1}{3}\%\)

3.38.2) Verify the BPI for \[15: 8, 6, 3, 1\]

BPI = \(\frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\) = \(37.5\%, 37.5\%, 12.5\%, 12.5\%\)

COMMENT

3.39) General Example: Consider the WVS \[7: 4, 3, 2, 1\]. Use A as the 4-vote player, B as the 3-vote player, C as the 2-vote player, D as the 1-vote player.

a) Which voter is a dictator? If no one is a dictator, write “none”

b) Which voter is a dummy voter? If no one is a dummy voter, write “none”

c) Which voter has veto power? If no one has veto power, write “none”
d) Compute the number of sequential coalitions.
e) Compute the number of voting coalitions.
f) Find the threshold value for the winning coalitions.
g) Find the threshold value for the blocking coalitions.
h) List all sequential coalitions in which C is pivotal.
i) List all blocking coalitions in which B is critical.
j) Compute SSPI(C)
k) Compute BPI(B)
l) Compute SSPI(D)
m) Compute BPI(D)
n) If A is critical in 5 winning coalitions, then BPI(A) = __?
o) If BPI(C) = 2, then C is critical in how many blocking coalitions?
p) List the minimal winning coalitions
q) List the minimal blocking coalitions
r) Which of the following WVS is equivalent to the given?
   Explain;  a) [19: 3, 6, 9, 19];  b) [33: 20, 14, 10, 6];
              c) [24: 18, 12, 8, 3]

ANSWERS: a) none; b) none; c) A; d) 4! = 24; e) 2^4 = 16; f) 7; g) 4; h) ADCB, DACB; i) {B,C}, {B,D}, {B,C,D};   j) 1/12;   k) 6; l) 0; m) 0; n) 10; o) 1; p) {A,B}, {A,C,D}; q) {A}, {B,C}, {B,D}; r) only b. Let the 20-vote player take the place of A, the 14-vote player takes the place of B, the 10-vote player takes the place of C, and the 6-vote player takes the place of D. Then the WVS in part b has the same corresponding winning coalitions as the given: {A,B}, {A,B,C}, {A,B,D}, {A,C,D}, {A,B,C,D}. The WVS in part a is a dictatorship, so it cannot be equivalent to the given. For the WVS in part c, let the 18-vote player take the place of A, the 12-vote player takes the place of B, the 8-vote player takes the place of C, and the 3-vote player takes the place of D. Then the winning coalitions of the WVS in part d are: {A,B}, {A,C}, A,B,C}, {A,B,D}, {A,C,D}, {A,B,C,D}, which are not the same as the corresponding winning coalitions of the given WVS.
SAMPLE QUESTIONS

1) Describe the system: a) [63: 60, 65]; b) [51: 40, 30, 20, 10]; c) [8: 3, 1, 1, 1, 1, 1, 1]

2) Four brothers and sisters (two sisters and two brothers) own a trust fund. Their parents assigned ten votes to each of the two sisters and five votes to each of the two brothers. The quota for passing a measure is 16 or more votes. Represent this weighted voting system mathematically.

3) Suppose a county commission consists of three members, one representing each of the three cities in the county. The commissioner from city A holds 49 votes, the commissioner from city B holds 40 votes, and the commissioner from city C holds 11 votes. If 60 votes are required to pass a measure, represent this weighted voting system mathematically.

4) Suppose that a small corporation consists of eight stockholders. Ann holds 200 votes, Sy holds 175 votes, Nancy holds 160 votes, Amal holds 150 votes, Ben holds 110 votes, Tyler has 80 votes, Angela holds 60 votes, and Josefa holds 100 votes. The eight members decide to form a Board of Directors to oversee the company and agree that it takes 700 or more votes to pass a motion. Represent this weighted voting system mathematically.

5) What is wrong with the WVS: [40: 20, 10, 9]?

6) What is wrong with the WVS: [21: 7, 5, 4, 3, 1]?

7) What is wrong with the WVS: [34: 30, 20, 10, 9]?

8) What is wrong with the WVS: [10: 8, 6, 4, 3]?

9) List all possible values of the quota q in the WVS: [q : 15, 12, 9, 5]

10) List all possible values of the quota q in the WVS: [q : 8, 8, 6, 5, 4, 4, 3, 1]

11) A weighted voting system consists of 5 voters with weights 9, 8, 6, 5, and 3. Find all possible values of the quota q.

12) A corporation has four partners with weights 8, 5, 3, and 2. The bylaws of the corporation require that two-thirds of the votes are needed to pass a motion. What is the value of the quota?

13) What are the values that q cannot have in the WVS [q : 50, 30, 20]?

14) In the WVS: [q : 10, 8, 4, 2, 1, 1], what are the values that q cannot have?

15) A weighted voting system consists of five players. Player 1 holds 5 votes, player 2 holds 4 votes, player 3 holds 3 votes, player 4 holds 2 votes, and player 5 holds 1 vote. Find all possible of the quota q.

16) Of the five members of a club, the president holds 5 votes, the v-p holds 4 votes, the treasurer holds 3 votes, the secretary holds 2 votes, and the assistant to the secretary holds 1 vote. List all possible values of the passing quota q.

17) A committee consists of seven members. The chairman holds 3 votes, the vice-chairman holds also 3 votes, and the rest of the members hold 1 vote each. What are the values that the passing quota q cannot have?
18) A committee consists of 5 faculty members, a provost, a dean, and a chairman. If the provost and the dean hold 10 votes each, the chairman holds 6 votes, and each of the faculty members holds 2 votes, find the values that q cannot have.

19) In a weighted voting system: a) what is a player? b) what is the weight? c) what is the quota? d) what is the range of values that the quota can have? e) what does it mean for a motion to pass? To not pass? In each part, do not give any examples… just explain in your own words.

20) Consider the WVS: [36: 16, 14, 10, 9]. Use A to represent the 16-vote player, B for the 14-vote player, C for the 10-vote player, D for the 9-vote player.
   a) Which voter is a dictator? If there are no dictators, write “none”
   b) Which one(s) has veto power? If no one has veto power, write “none”
   c) Which one(s) is a dummy voter? If no one is dummy voter, write “none”

21) Consider the WVS: [25: 15, 10, 5, 4]. Use A to represent the 15-vote player, B for the 10-vote player, C for the 5-vote player, D for the 4-vote player.
   a) Which voter is a dictator? If there are no dictators, write “none”
   b) Which one(s) has veto power? If no one has veto power, write “none”
   c) Which one(s) is a dummy voter? If no one is dummy voter, write “none”

22) Consider the following weighted voting systems: a) [6: 5, 4, 3]; b) [4: 3, 2, 1];
    c) [7: 8, 3, 1, 1]; d) [12: 5, 4, 3, 2]; e) [5: 4, 3, 2]; f) [15: 7, 6, 3]
    i) Which systems have a dictator? (ii) Which systems are invalid?
    (iii) Which systems have voters with veto power?
    (iv) Which systems have dummy voters?
    (v) Which systems have more than one voter with veto power?

23) Consider the weighted voting system [16: 10, 8, 5, 2]
   a) Which voters are dictators (if any)?
   b) Which voters are dummy voters (if any)?
   c) Which voters have veto power (if any)?

24) Answer “True” or “False” in each of the following: a) In weighted voting systems, voters may hold or control unequal weights; b) the preferences of some voters may carry more weight than the preferences of other voters; c) a player with a larger weight always has more voting power that a player with a smaller weight; d) one single voter may carry all the power to influence a decision.

25) Answer “True” or “False” in each of the following: a) In weighted voting systems, it is possible to have two dictators; b) if there is a dictator, then the dictator has all the power; c) dummy voters carry no voting power; d) it is possible to have two dummy voters; e) every voter with veto power is a dictator; f) every dictator has veto power; g) it is possible that in a WVS two or more voters have veto power; h) it is possible that in a WVS all voters have veto power; i) it is possible that in a WVS no voters have veto power; j) it is possible that in a WVS all voters are dummy voters.
26) In a weighted voting system, a resolution is guaranteed pass if: a) the system has a voter with veto power and this voter favors the resolution; b) the system has a dictator and the dictator favors the resolution; c) the system has a dummy voter and s/he favors the resolution; d) the system has a group of voters who favor the resolution; e) the voters who favor the resolution have a total combined weight equal to or greater than the quota.

27) a) Find the number of sequential coalitions in the WVS: [65: 62, 66]
   b) List all possible sequential coalitions in the WVS [65: 62, 66]
      Use A for the 62-vote player; B for the 66-vote player.

28) a) Find the number of sequential coalitions in the WVS: [12: 7, 6, 4, 1]
   b) List all possible sequential coalitions in the WVS: [12: 7, 6, 4, 1]
      Use A for the 7-vote player; B for the 6-vote player; C for the 4-vote player; D for the 1-vote player.

29) a) Find the number of sequential coalitions in the WVS: [6: 4, 3, 2]
   b) List all possible sequential coalitions in the WVS: [6: 4, 3, 2]
      Use A for the 4-vote player; B for the 3-vote player; C for the 2-vote player

30) a) Explain what a sequential coalition is (no examples); b) Explain what a pivotal voter is (no examples).

31) a) Compute the System SSPI for each WVS: [65: 62, 66]
   b) Compute the System SSPI for each WVS: [6: 4, 3, 2]
   c) Compute the System SSPI for each WVS: [12: 7, 6, 4, 1]

32) Correctly state the Paradox of New Members. No examples, no diagrams.

33) Suppose that a WVS originally consisted of four members and they shared equal voting power (25% each). Suppose that after some time, a new member joined the system. If the quota remained unchanged and the five members also shared equal voting power (now 20% each), does this illustrate the occurrence of the Paradox of New Members? Explain.

34) Suppose that a WVS originally had four members and they shared equal voting power with System SSPI: (A: 25%; B: 25%; C: 25%; D: 25%). After some time, a new member, E, joined the system. If the quota remained unchanged and the new System SSPI was (A: 24%; B: 19%; C: 19%; D: 19%; E: 19%), does this illustrate the occurrence of the Paradox of New Members? Explain.

35) Suppose that a WVS originally had five members, \{A, B, C, D, E\} with System SSPI: \(\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)\). After some time, a new member, F, joined the system. If the quota remained unchanged and the new System SSPI was \(\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{6}, \frac{1}{10}, \frac{1}{120}\right)\), does this illustrate the occurrence of the Paradox of New Members? Explain.
36) If a WVS has SSPI=$\left(\frac{3}{8}, \frac{2}{5}, \frac{1}{12}, x, \frac{2}{15}\right)$, find the value of x.

37) Find the SSPI for the WVS: [6: 3, 1, 1, 1, 1, 1, 1]
38) Find the SSPI for the WVS: [23: 7, 4, 4, 4, 4, 4, 4, 4, 4]
39) Explain the (main) difference between a pivotal voter and a critical voter.

40) Compute the number of coalitions in the WVS: [7: 6, 4, 2]. List the winning, blocking, and losing coalitions. Use set braces {} for each coalition. Let A represent the 6-vote player, B is the 4-vote player, C is the 2-vote player.
41) Compute the number of coalitions in the WVS: [9: 6, 4, 3, 2]. List the winning, blocking, and losing coalitions. Use set braces {} for each coalition. Let A represent the 6-vote player, B is the 4-vote player, C is the 3-vote player, D is the 2-vote player.
42) Consider the WVS [14: 8, 6, 5, 4]. If A represents the 8-vote player, B is the 6-vote player, C is the 5-vote player, D is the 4-vote player: a) find the critical voters in the winning coalition {A, B, C}; b) find the critical voters in the blocking coalition {B, C, D}.
43) Consider the WVS: [16: 9, 7, 6, 4]. Use A for the 9-vote player, B is the 7-vote player, C is the 6-vote player, D is the 4-vote player: a) Is \{A,C,D\} a MWC? b) Is \{A,B,D\} a MWC? c) Is \{B,D\} a MBC? d) Is \{A,B,D\} a MBC?
44) Find the BPI for the WVS [7: 5, 3, 1]. Use A for the 5-vote player, B for the 3-vote player, C for the 1-vote player. Use set braces correctly. Points will be deducted for not using correct set brace notation.
45) Find the BPI for the WVS [10: 7, 5, 3, 2]. Use A for the 7-vote player, B for the 5-vote player, C for the 3-vote player, D for the 2-vote player. Use set braces correctly. Points will be deducted for not using correct set brace notation.
46) Find the BPI for the WVS [7: 6, 2, 2, 2]. Use A for the 6-vote player, B for the first 2-vote player, C for the second 2-vote player, D for the third 2-vote player. Use set braces correctly. Points will be deducted for not using correct set braces.
47) Consider the WVS: [35: 12, 10, 9, 7, 6, 5, 4, 3, 2, 2, 2] a) Find the number of possible subsets of voters b) Find the number of possible sequential coalitions. c) Find the threshold value of a winning coalition. d) Find the threshold value of a blocking coalition.
48) (BONUS) Compute the BPI: a) [10: 5,2,2,2,2,2,2,2]; b) [17: 5,5,3,3,3,3,3,3,3,3]
49) In each part, classify the given WVS as Dictatorship, Clique, Majority, Chair veto, or Consensus: a) [4: 2, 2, 1]; b) [3: 1, 1, 1]; c) [3: 2, 1, 1]; d) [2: 1, 1, 1]
50) In each part, classify the given WVS as Dictatorship, Clique, Majority, Chair veto, or Unanimity: a) [35: 20, 10, 5]; b) [4: 4, 1, 1]; c) [10: 12, 5, 2]; d) [18: 8, 6, 5]
51) In each part, classify the given WVS as Dictatorship, Clique, Majority, Chair veto, Unanimity: a) [50: 18,17,16]; b) [20:22,10,7]; c) [11:8,6,5]; d) [10:7,6,2]
52) In each part, classify the given WVS as Dictatorship, Clique, Majority, Chair veto, Unanimity: a) [14: 9, 6, 5]; b) [33: 20, 18, 15]; c) [18: 10, 8, 7]; d) [10: 5, 4, 2]

53) Show that the WVS's [4: 2, 2, 1] and [10: 7, 6, 4] are not equivalent.
54) Show that the WVS's [12: 8, 6, 4, 2] and [20: 13, 10, 7, 3] are equivalent.

55) Consider the weighted voting system [23: 16, 8, 4, 2]
   a) Which voters are dictators? If there are no dictators, write “none”; b) Which voters have veto power? If no voter has veto power, write “none”; c) Which voters are dummy voters? If there are none, write “none”; d) Is this system a four-member consensus? Justify your answer; e) How many sequential coalitions does this system have? f) List the permutations where the 8-vote player is pivotal. If there are none, write “none”; g) List the permutations where the 4-vote player is pivotal. If there are none, write “none”; h) How many winning coalitions (WC) does this system have? i) Give the threshold value of the WC’s; j) Give the threshold value of the blocking coalitions (BC’s); k) List the WC’s where the 16-vote player is critical. If there are none, write “none”; m) Compute the SSPI of the 8-vote player; n) Compute the SSPI of the 4-vote player; o) Compute the SSPI of the 16-vote player; p) Compute the BPI of the 16-vote player; q) Compute the BPI of the 2-vote player.

56) Suppose that a WVS originally consisted of four members and the system BPS = (2, 2, 2, 2). Suppose that after some time, a new member joined the system. If the quota remained unchanged and the new BPI = (8, 8, 8, 8, 8), does this illustrate the occurrence of the Paradox of New Members? Explain.

57) Consider the WVS: [10: 6, 4, 3]; a) classify the system as a dictatorship, a unanimity, majority, clique, or a chair veto; b) compute the system BPS; c) suppose that after some time, a new member with weight 1 joined the system, so the new system became: [10: 6, 4, 3, 1]. If the quota remained unchanged, compute the BPI of the new system; d) does this illustrate the occurrence of the Paradox of New Members? Explain.

57) a) clique; b) (4, 4, 0); c) (10, 6, 2, 2); d) Yes. Originally, the 3-vote player was a dummy voter (so her voting power was 0%). After the new member joined, the voting power of the 3-vote player increased to 10% (not a dummy voter any more)

58) Consider the WVS: [15: 12, 8, 4, 2]. Let A represent the 12-vote player, B is the 8-vote player, C is the 4-vote player, D is the 2-vote player; a) Which voter is a dictator? If there is no dictator, write “none”; b) Which voter is a dummy voter? If there is no dummy voter, write “none”; c) Which voter has veto power? If no voter has veto power, write “none”; d) Compute the number of sequential coalitions; e) Compute the number of voting coalitions; f) Find the threshold value for the winning coalitions; g) Find the threshold value for the blocking
coalitions; h) List all sequential coalitions in which B is pivotal; i) List all blocking coalitions in which C is critical; j) Compute SSPI(B); k) Compute BPI(C).

: [15: 12, 8, 4, 2].

i) List all blocking coalitions in which B is critical.

j) Compute SSPI(C)
k) Compute BPI(B)
l) Compute SSPI(D)
m) Compute BPI(D)
n) If A is critical in 5 winning coalitions, then BPI(A) = __?
o) If BPI(C) = 2, then C is critical in how many blocking coalitions?
p) List the minimal winning coalitions
q) List the minimal blocking coalitions
r) Which of the following WVS is equivalent to the given?
   Explain; a) [19: 3, 6, 9, 19]; b) [33: 20, 14, 10, 6];
c) [24: 18, 12, 8, 3]

ANSWERS: a) None; b) D; c) A; d) 4! = 24; e) 2^4 = 16; f) 15; g) W-q+1 = 12;
h) ABCD; ABDC; ADBC; DABC; i) {B,C}; {B,C,D}; j) 1/6; k) 4

ANSWERS TO SELECTED SAMPLE QUESTIONS

1) [63: 60, 65] is described as a weighted voting system with quota 63 and two voters, one holding a weight of 60 votes and the other holding a weight of 65 votes. A resolution passes whenever the total number of votes of all voters who favor the resolution is 63 or more.
2) [16: 10,10,5,5]; 3) [60: 49,40,11]; 4) [700: 200,175,160,150,110,80,60,100]
5) The quota, q=40, is more than the sum of the weights of all voters (39), so no resolution can ever pass.
6) No motion would ever pass since the quota, q=21, is more than the total weight of all voters (20).
7) The quota, q=34, is less than half the sum of the weights of all voters (69), so ridiculous resolutions may pass.
8) The quota, q=10, is less than half of the total weight of all voters (21) , so ridiculous resolutions may pass.
9) 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41
10) 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39
11) 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31; 12) 12
13) q cannot be 50 or less than 50 and q cannot be more than 100
14) q cannot be 14 or less than 14 and $q$ cannot be more than 28.
15) 8, 9, 10, 11, 12, 13, 14, 15; 16) 8, 9, 10, 11, 12, 13, 14, 15
17) q cannot be 5 or less than 5 and $q$ cannot be more than 11.
18) q cannot be less than 19 and $q$ cannot be more than 36
20) a) none; b) A, B; c) none; 21) a) none; b) A, B; c) C, D
22) (i) c; (ii) a; (iii) b; c; d; f; (iv) c; d; (v) d; f
23) (a) none; (b) none; (c) the voter with weight 10
24) a) true; b) true; c) false; d) true
25) a) false; b) none; c) true; d) true; e) false; f) true; g) true; h) true; i) true; j) false
26) b, e; 27) a) 2; b) AB, BA
28) a) 24; b) ABCD, ABDC, ACBD, ACDB, DABC, BACD, BCDA, BDAC, BDCA, CABD, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA
29) a) 6; b) ABC, ACB, BAC, BCA, CAB, CBA
31) a) $(1, 0)$; b) $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$; c) $\left(\frac{7}{12}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}\right)$
33) No. The voting power did not increase for any of the original four members.
34) No. The voting power did not increase for any of the original four members.
35) Yes. The voting power of player A increased from $\frac{1}{3}$, (33 $\frac{1}{3}$ %), to $\frac{2}{5}$, (40%). Also, the voting power of player B increased from $\frac{1}{6}$, (16 $\frac{2}{3}$ %), to $\frac{1}{5}$, (20%).
36) $x = \frac{1}{120}$; 37) $\left(\frac{3}{7}, \frac{2}{21}, \frac{2}{21}, \frac{2}{21}, \frac{2}{21}, \frac{2}{21}\right)$; 38) $\left(\frac{1}{5}, \frac{4}{45}, \frac{4}{45}, \frac{4}{45}, \frac{4}{45}, \frac{4}{45}, \frac{4}{45}, \frac{4}{45}\right)$
40) There are 8 coalitions in total. Winning Coalitions: {A,B}, {A,C}, {A,B,C}; Blocking Coalitions: {A}, {A,B}, {A,C}, {B,C}, {A,B,C}; Losing Coalitions: { }, {B}, {C}
42) a) A, B; b) only B. 43) a) Yes, all voters are critical; b) No. D is not critical; c) Yes, both voters are critical; d) No. None of the voters are critical.
44) (4, 4, 0). 45) (10, 6, 6, 2). 46) (14, 2, 2, 2).
47) a) 2048; b) 39,916,800; c) 35; d) 28
49) a) clique; b) consensus; c) chair veto; d) majority
50) a) unanimity; b) dictatorship; c) dictatorship; d) unanimity
51) a) unanimity; b) dictatorship; c) majority; d) clique
52) a) chair veto; b) majority; c) clique; d) unanimity
53) In the first WVS, let A represent the first 2-vote player, B is the second 2-vote player, C is the 1-vote player. This system is a clique and the winning coalitions are {A,B}, {A,B,C}. In the second WVS, let A represent the 7-vote player, B is
the 6-vote player, C is the 4-vote player. This system is a majority and winning coalitions are \{A,B\}, \{A,C\}, \{B,C\}, \{A,BC\}. Clearly, the two systems do not have the same corresponding winning coalitions and therefore, they are not equivalent WVS’s.

54) In the first WVS, let A represent the 8-vote player, B is the 6-vote player, C is the 4-vote player, D is the 2-vote player. The winning coalitions are \{A,B\}, \{A,C\}, \{A,B,C\}, \{A,B,D\}, \{A,C,D\}, \{B,C,D\}, \{A,B,C,D\}. In the second WVS, if we let the corresponding voters take the places of A as the 13-vote player, B is the 10-vote player, C is the 7-vote player, D is the 3-vote player, then the winning coalitions are the same. This system is a majority and its MWC’s are \{A,B\}, \{A,C\}, \{A,B,C\}, \{A,B,D\}, \{A,C,D\}, \{B,C,D\}, \{A,B,C,D\}. Clearly, the two systems have the same corresponding winning coalitions and therefore, they are equivalent WVS’s.

55) No, originally, the voting power was 25% for each of the four members. After the new member joined, the new voting power was 20% for each of the five members. So, none of the original members had an increase in voting power.

SAMPLE QUESTIONS

[27] a) In the weighted voting system \[100: 98, 2\], if A is the 98-vote player, verify that the only minimal winning coalition is \{A,B\}. Name the system.
   b) In the weighted voting system \[52: 49, 51\], if A is the 49-vote player, verify that the only minimal winning coalition is \{A,B\}. Name the system.
   c) Explain why the systems in a), b) are equivalent.
   d) In the weighted voting system \[100: 100, 50\], if A is the 100-vote player, verify that the minimal winning coalition is \{A\}. Name the system.
   e) In the weighted voting system \[105: 90, 110\], if B is the 110-vote player, verify that the minimal winning coalition is \{B\}. Name the system.
   f) Explain why the systems in d), e) are equivalent.

(28) Use winning coalitions to show that \[2: 2, 1\]; \[5: 6, 3\]; \[60: 50, 65\] are equivalent systems. Compute the BPI and the SSPI and name the systems.

(29) Use winning coalitions to show that \[3: 3, 1, 1\]; \[8: 10, 3, 2\]; \[20: 5, 10, 20\] are equivalent systems. Compute the BPI and SSPI and name the systems.

(30) Use winning coalitions to show that \[3: 2, 1\]; \[8: 4, 4\]; \[30: 1, 29\] are equivalent systems. Compute the BPI and the SSPI and name the systems.

(31) Use winning coalitions to show that \[4: 2, 1, 1\]; \[15: 5, 5, 5\]; \[40: 10, 13, 18\] are equivalent systems. Compute the BPI and SSPI and name the systems.
(32) Use winning coalitions to show that $[6: 3, 3, 2]; [18: 10, 8, 7]; [60: 1, 20, 50]$ are equivalent systems. Compute the BPI and SSPI and name the systems.

(33) Use winning coalitions to show that $[5: 3, 3, 2]; [18: 9, 9, 9]; [100: 50, 60, 70]$ are equivalent systems. Compute the BPI and SSPI and name the systems.

(34) Use winning coalitions to show that $[5: 3, 2, 2]; [18: 10, 9, 8]; [50: 10, 30, 40]$ are equivalent systems. Compute the BPI and SSPI and name the systems.

(35) a) Find two distinct weighted voting systems equivalent to $[9: 8, 7]$
   b) Find two distinct weighted voting systems equivalent to $[9: 5, 11]$
   c) Find two distinct weighted voting systems equivalent to $[9: 9, 4, 2]$
   d) Find two distinct weighted voting systems equivalent to $[9: 6, 4, 1]$
   e) Find two distinct weighted voting systems equivalent to $[9: 6, 5, 4]$
   f) Find two distinct weighted voting systems equivalent to $[9: 8, 5, 2]$
   g) Find two distinct weighted voting systems equivalent to $[9: 1, 2, 6]$

(36) Explain the difference between a three-member clique and a three-member majority system.

(37) In which of the five types of three-member systems do all voters have equal power? Equal BPI? Equal SSPI?

[41] NEW TEXT-BOOK
   (a) Page 361: Review Vocabulary (all).
   (b) Page 362: Skills Check (all).
   (c) Pages 362-367: #1, 3, 4 (a: the weight-33 voter; b: 68; c: no), 6 (no, because the SSPI would not be zero), 7, 8 (a: BACD, BADC, BCAD, BDAC, CABD, CADB, CBAD, CDAB, DABC, DACB, DBAC, DCAB; b: ABCD, ABDC, CDBA, DCBA; c: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$), 9, 10 (a: $(2,2,2,2,2)$ or $(2,2,2,3,2,2)$; b: 240; c: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$); 13, 14 (may omit extra votes and just find the BPI; a: (4,0); b: (4,4,4); c: (6,2,2); d: (8,8,8,0); e: (12,4,4,4)); 15 (may omit the extra-vote part, but find the BPI); 17; 18 (a: 20; b: 4950; c: 4950; d: 126); 25; 26 (BPI=$(28,4,4,4,4)$); 27; 32; 35; 39; 40; 41

ANSWERS TO SELECTED SAMPLE QUESTIONS
[26] a) 32; b) 120; c) 14; d) 8; e) 3; f) 1
[27] a) consensus; b) consensus; c) they are 2-member consensus; d) dictatorship; e) dictatorship; f) they are 2-member dictatorships.
[28] BPI = (4, 0); SSPI = (1, 0); [29] BPI = (8, 0, 0); SSPI = (1, 0, 0)
[30] BPI = (2, 2); SSPI = $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; [38] a) (v); b) (i); c) (iv); d) (iii); e) (ii)
[39] a) (iii); b) none; c) none; d) (ii); e) (i) and (iv)