MTG 1204 Geometry for Educators **Professor Orr**
Notes on Section 2.1

Undefined Terms, Definitions, and Postulates

Euclidean geometry is what we call a "formal axiomatic system". This is a body of knowledge that is built very carefully and precisely, starting with a few undefined terms, a variety of additional terms that are each carefully defined, and a small set of "postulates". These are summarized below.

Undefined terms

It is necessary for an axiomatic system to have a starting point. We would like to be able to precisely define every term we use in our system, but this is impossible, because whatever words we might use to define each term must in turn be defined using other words, which must in turned be defined, and so on, ad infinitum.

Thus, we begin with three terms that are completely undefined. They are:

U1. *point*
U2. *line*
U3. *plane*

Now we can write some words to help us visualize what these terms mean, but it's important to realize that these words do not constitute *definitions*.

U1. Think of a *point* as a small dot that has been shrunk down so small that it has no size at all.

U2. Think of a *line* as a wire that has been stretched as tightly as possible, so that it is infinitesimally thin, but also perfectly straight and extending indefinitely in both directions.

U3. Think of a *plane* as a sheet of paper so thin that it has no thickness at all, and extending infinitely in all directions.

Notice that all of the other terms we will use in our system will either be defined in terms of these three undefined terms, or else be based upon our pre-existing understanding of the set of real numbers from arithmetic and algebra.

Definitions

We state here a few preliminary definitions. Many more will follow as our formal system unfolds.

D1. *Space* is the set of *all* points.

D2. A *geometric figure* is any collection of points.

D3. Two or more points are said to be *collinear* if there is one line that contains all of them.

D4. Two or more points are *coplanar* if there is one plane that contains all of them.

We will state additional definitions as we go along, and as the need arises. But we now turn our attention to the set of *postulates* that we need as the foundation of our formal system.

Postulates

A postulate is a statement that is taken to be true without question or proof. Once we have accepted a short list of postulates, we will be in a position to use deductive logic to *prove* a great many other propositions that follow from the postulates.

P1. Every line contains at least two distinct points.

P2. Given two distinct points, there is one and only one line containing both of them.

P3. If two distinct points lie in a given plane, then the line containing them is also in the plane.

P4. Given three noncollinear points, there is one and only one plane containing them; conversely, given a plane, there are at least three noncollinear points lying in that plane.

P5. In space, there exist at least four points that are not all coplanar.

P6. (The "Ruler Postulate") Every line can be made into an exact copy of the real number line.

Here are a few comments about the postulates.

P1 ensures that every line contains more than one point.

P2 ensures that every line is "straight", because if lines were allowed to curve, then it would be possible to find more than one line passing through two given points.

P3 likewise ensures that every plane is "flat", because if planes were allowed to curve, then the (straight) line through two given points would not be able to stay in the same plane through its entire length.

P4, together with P2, ensures that every plane contains more than one line.

P5 ensures that space contains more than one plane.

P6 opens up an entire realm of possibilities. We assume a prior familiarity with the set of real numbers and the way they are ordered. P6 implies that there are in fact not just two points on a given line, but in fact *infinitely many* points, and that they are arranged in an orderly fashion just like real numbers. Because of the so-called "completeness property" of the real numbers (which says that between any two real numbers there lies another real number), P6 also guarantees that between any two points on a given line there is always another point.

Given a line, one may choose a point to correspond to the real number 0 and another point to correspond to the real number 1. Once these two points are chosen (and they exist by P1), every other point on the line is automatically made to correspond to a unique real number. Thus, P6 allows us to bring in the notion of "distance", which is crucial to much of what goes on in geometry.

More definitions

We now need to define some more terms.

D5. The real number corresponding to a point on a line is called the ***coordinate*** of the point.

D6. The ***distance*** between points A and B is the nonnegative difference in their coordinates.

D7. Given points A and B on a line, the ***line segment*** $\overline{AB}$ is the set of all points on the line whose coordinates are between those of A and B.

D8. Points A and B are called the ***endpoints*** of segment $\overline{AB}$.

D9. The *length* of segment $\overline{AB}$ is the distance between A and B, and is denoted by AB.

D10. Two segments are *congruent* if they have the same length. In symbols, we will write $\overline{AB} \cong \overline{CD}$ to indicate that segment $\overline{AB}$ is congruent to segment $\overline{CD}$. It follows readily that when $\overline{AB} \cong \overline{CD}$, we also have $AB = CD$ — that is, when two segments are congruent, their lengths are equal.

D11. The *ray* $\overrightarrow{AB}$ is composed of points A and B, together with all points X on line $\overleftrightarrow{AB}$ such that either X lies between A and B or B lies between A and X. More simply stated, a *ray* is a half-line with an endpoint.

D12. An *angle* is the union of two rays or two line segments sharing a common endpoint. That common endpoint is call the *vertex* of the angle, and the rays or segments themselves are called the *sides* of the angle. When segments $\overline{AB}$ and $\overline{AC}$ form an angle, we may denote it by $\angle BAC$ or by $\angle CAB$. If it is clear which angle is meant, we may also denote an angle by $\angle A$, where A is the vertex.

D13. If a ray or segment is rotated in such a way that its endpoint remains fixed and every other point on it traverses a full circle counterclockwise, the angle so formed is called 1 *revolution*.

D14. The measure of an angle that is $\frac{1}{360}$ of a complete revolution is 1 *degree*, denoted by 1°.

D15. A *protractor* is an instrument in the shape of a semi-circular region, with one straight edge whose midpoint is marked, and with the semi-circular border marked off at regular intervals.

In order to proceed further, we find it advisable to adopt one additional postulate, as follows.

P7. (The "Protractor Postulate") If one ray of an angle is placed at 0° on a protractor and the vertex is placed at the midpoint of the bottom edge, then there is a 1-1 correspondence between all other rays that can serve as the second side of the angle and the real numbers between 0° and 180° inclusive, as indicated by a protractor.

In essence, P7 says that every angle can be *measured* in degrees by a real number between 0 and 180 inclusive; and vice-versa, for every such number there is an angle having that measure. Note that although it is certainly possibly to form an angle whose measure is greater than $180°$, such an angle will always have the same terminal side as some angle whose measure is between $0°$ and $180°$ inclusive. For example, an angle of $210°$ may also be regarded as an angle of $150°$ by simply reversing the direction of the rotation.

Now, more definitions.

D16. If the vertex of an angle is placed at the center of the bottom of the protractor and one ray of the angle is placed at $0°$ at either end of the protractor, then the *measure of the angle* is given by the number that falls on the other ray. The measure of $\angle BAC$ is denoted by the symbol m$(\angle BAC)$.

(Note: It is not a good thing to confuse an angle with the *measure* of that angle. After all, an angle is the union of two rays or two line segments, whereas the measure of the angle is a real number of degrees. We nevertheless tend to speak as if they were one and the same. For example, if m$(\angle BAC) = 45°$, we will often write simply $\angle BAC = 45°$.)

D17. An *acute angle* is one whose measure is between $0°$ and $90°$.

D18. A *right angle* is one whose measure is exactly $90°$.

D19. An *obtuse angle* is one whose measure is between $90°$ and $180°$.

D20. A *straight angle* is one whose measure is exactly $180°$.

D21. A *reflex angle* is one whose measure is between $180°$ and $360°$.

D22. Two lines that form a right angle are said to be *perpendicular*. If lines l and m are perpendicular, we write $l \perp m$.

D23. Two angles are said to be *congruent* if they have the same measure. For example, we may write $\angle BAC \cong \angle EDF$ to mean that m$(\angle BAC) =$ m$(\angle EDF)$.

D24. Two angles are said to be *complementary* if the sum of their measures is $90°$.

D25. Two angles are said to be *supplementary* if the sum of their measures is $180°$.

D26. Two angles are said to be *adjacent* if they share a common vertex and a common side, and the common side lies between the other two sides. (The last clause in the definition ensures that the angles do not "overlap".)

Note: An immediate consequence of D26 and D16 is that when two angles are adjacent, the sum of their measures gives the measure of the angle formed by the two sides the angles do *not* have in common. For example, suppose $\angle BAC$ and $\angle CAD$ are adjacent with $\angle BAC = 30°$ and $\angle CAD = 20°$. Then it follows that $\angle BAD = 50°$.

Notes on Section 2.2

Polygons and Circles

D27. A *simple closed curve* is a figure that lies in a plane and can be traced so that the starting and ending points are the same and no part of the curve is crossed or retraced.

D28. A *circle* is a simple closed curve that consists of the set of all points equidistant from a given point, called the *center* of the circle.

D29. A line segment joining the center of a circle with a point on the circle is called a *radius* of the circle. The length of a radius is referred to as *the radius* of the circle.

D30. A line segment containing the center of a circle and with endpoints on the circle is called a *diameter* of the circle. The length of a diameter is referred to as *the diameter* of the circle.

D31. A *polygon* is a simple closed curve composed of line segments.

D32. A *triangle* is a polygon with three sides.

D33. A *quadrilateral* is a polygon with four sides.

D34. A *pentagon* is a polygon with five sides.

D35. A *hexagon* is a polygon with six sides.

D36. An *octagon* is a polygon with eight sides.

D37. A *decagon* is a polygon with ten sides.

D38. An *n-gon* is a polygon with n sides, where n is any integer greater than or equal to 3.

Fact: Given three noncollinear points $A, B,$ and $C,$ there is a unique triangle having these points as vertices. The triangle is denoted by $\triangle ABC$ and the line segments $\overline{AB}, \overline{BC},$ and $\overline{AC}$ are referred to as the *sides* of the triangle. Each pair of sides forms an angle, so every triangle has three angles as well as three sides.

D39. A triangle is *equilateral* if all three of its sides are congruent.

D40. A triangle is *isosceles* if at least two of its sides are congruent.

D41. A triangle is *scalene* if no two of its sides are congruent.

D42. A triangle is *acute* if all three of its angles are acute.

D43. A triangle is *right* if one of its angles is a right angle.

D44. A triangle is *obtuse* if one of its angles is obtuse.

D45. A triangle is *equiangular* if all three of its angles are congruent.

D46. In an isosceles triangle, the angles opposite the congruent sides are called *base angles*, and the side between these two angles is called the *base.* The angle between the congruent sides is called the *vertex angle*.

D47. In a right triangle, the side opposite the right angle is called the *hypotenuse* and the two shorter sides are called the *legs*.

D48. Two lines are said to be *parallel* if they lie in the same plane and do not intersect (i.e. they do not have a point in common). When line l is parallel to line m, we write $l \parallel m$.

D49. A *square* is a quadrilateral with all sides congruent and all angles right.

D50. A *rectangle* is a quadrilateral with all angles right.

D51. A *rhombus* is a quadrilateral with all sides congruent.

D52. A *parallelogram* is a quadrilateral with opposite sides parallel.

D53. A *trapezoid* is a quadrilateral with exactly one pair of sides parallel.

D54. An *isosceles trapezoid* is a trapezoid with the two nonparallel sides congruent.

D55. A *kite* is a quadrilateral with two pairs of adjacent sides that are congruent and nonoverlapping.

D56. A geometric figure has *reflection symmetry* if there is a line along which the figure may be folded so that one half of the figure matches exactly with the other half. The fold line is called the *line of symmetry* or the *axis of symmetry*. Note that a figure may have more than one axis of symmetry.

D57. A geometric figure has *rotation symmetry* if it can be rotated about a point less than a full turn, so that the resulting image is identical to to original figure. The point is called the *center of rotation symmetry*.

Notes on Section 2.3

Angle Measure in Polygons and Tessellations

D58. A *polygonal region* is a polygon together with the portion of the plane that is enclosed by the polygon.

D59. A *tessellation* (also known as a *tiling*) is an arrangement of polygons that completely covers a plane without overlap.

Any triangle can be used to tessellate the plane. Such a tessellation provides evidence to support the following theorem. Note that a *theorem* differs from a postulate in that a theorem is a statement that can be **proved**. We will begin using deductive logic to prove theorems later in this course. But for now the following theorem is stated without proof.

T1. The sum of the measures of the angles in a triangle is $180°$.

D60. An angle formed by two adjacent sides of a polygon is called a *vertex angle* of the polygon.

D61. A line segment joining two nonadjacent vertices of a polygon is called a *diagonal* of the polygon.

Given any polygon with more than three sides, we may subdivide it into triangles by constructing all possible diagonals from any one vertex. Now there are exactly $n-3$ diagonals emanating from any one vertex of an n-gon, and a few examples will show that these $n-3$ diagonals, together with the two sides meeting at that vertex, form a total of $n-2$ triangles. It is therefore easy to believe the following theorem, which we state without formal proof at this time.

T2. The sum of the measures of the vertex angles in an n-gon is $(n-2) \cdot 180°$.

D62. A *regular polygon* is a polygon in which all sides are congruent and all vertex angles are congruent.

The following theorem is an immediate consequence of T2 in combination with the above definition.

T3. The measure of a vertex angle in a regular n-gon is $\frac{(n-2) \cdot 180°}{n}$.

D63. A *regular tessellation* is a tessellation composed of regular polygons that are all the same size and shape.

Since a regular polygon is equiangular, and since the polygonal regions used in a regular tessellation must fit together exactly around each vertex, it follows that the measure of each vertex angle must divide evenly into 360°. And from this it follows in turn that **the only possible regular tessellations of the plane are composed of equilateral triangles, squares, or regular hexagons.**

Notes on Section 2.4

Three-Dimensional Shapes

D64. A *polyhedron* is a three-dimensional shape composed of polygonal regions any two of which have at most a common side. Moreover, it is an enclosed, connected finite portion of space without holes. (Note that the plural of polyhedron is *polyhedra*.)

The polygonal regions forming the boundary of a polyhedron are called its *faces*, the line segments those faces have in common are called its *edges*, and a point where two edges meet is called a *vertex*.

D65. A *prism* is a polyhedron with two opposite faces, called *bases*, that are identical polygonal regions in parallel planes. The other faces of a prism are called *lateral faces*.

Note that prisms are generally named according to the shapes of their bases. For example, we have triangular prisms, rectangular prisms, hexagonal prisms, etc.

D66. A *right prism* is a prism all of whose lateral faces are rectangles. A prism that is not right is called an *oblique prism*.

D67. A *pyramid* is a polyhedron consisting of a polygonal region for its *base* and triangular regions as *lateral faces*. The single vertex that is shared by all of the lateral faces is called the *apex* of the pyramid.

Like prisms, pyramids are named according to the shapes of their bases. So we find triangular pyramids, square pyramids, etc.

D68. A *regular pyramid* is one whose base is a regular polygon. If the faces of a regular pyramid are *isosceles* triangles, then it is called a *right regular pyramid*, and the height of any of those isosceles triangular faces is called the *slant height* of the pyramid. Otherwise, a regular pyramid whose faces are not isosceles is known as an *oblique regular pyramid*.

D69. A *regular polyhedron* is a polyhedron all of whose faces are regular polygons of exactly the same size and shape.

The Greeks discovered that only five regular polyhedra exist. They are named as follows:

D70. A *tetrahedron* is a polyhedron with four faces, all of which are equilateral triangles. (Note that this is a special case of the right regular triangular pyramid.)

D71. A *hexahedron* (more commonly known as a *cube*) is a polyhedron with six faces, all of which are squares. (Note that this is a special case of the right square prism.)

D72. An *octahedron* is a polyhedron with eight faces, all of which are equilateral triangles. (Note that an octahedron consists of two right square pyramids with a common base.)

D73. A *dodecahedron* is a polyhedron with twelve faces, all of which are regular pentagons.

D74. An *icosahedron* is a polyhedron with twenty faces, all of which are equilateral triangles.

Be sure to notice the table on page 78 of your text listing the number of faces (F), the number of vertices (V), and the number of edges (E) possessed by each of the regular polyhedra. They exhibit the pattern known as **Euler's formula**—namely that $F + V = E + 2$. This relationship is in fact borne out by *all* polyhedra, not just by regular polyhedra.

D75. A *semiregular polyhedron* is one whose faces are regular polygons of two or more types, but with the same arrangement of polygons at each vertex. (A soccer ball is a classic example, having faces that are all either regular pentagons or regular hexagons, with each vertex shared by two hexagons and one pentagon.)

D76. A *circular cylinder* is a three-dimensional shape with two identical circular *bases* in parallel planes and a lateral surface formed by all of the line segments joining corresponding points on the two circles. When the line segments are perpendicular to the planes containing the two bases, we have a *right circular cylinder*; otherwise we have an *oblique circular cylinder*.

D77. A *circular cone* is a three-dimensional shape formed by a circular *base* together with the line segments joining each point on the circle with a single point, called the *apex*, not lying in the same plance as the base.

If the apex lies on a line perpendicular to the base at its center, then the cone is called a *right circular cone*. Otherwise the cone is called *oblique*. In a right circular cone, the distance between the apex and the center of the base is called the *height* of the cone, and the distance between the apex and any point on the circular edge of the base is called the *slant height*.

D78. A *sphere* is the set of all points in space that are at a fixed distance from a given *center* point. Any line segment joining the center to a point on the sphere is called a *radius* of the sphere; by definition, all radii have the same length, and that length is called *the radius* of the sphere. Likewise, any line segment containing the center whose endpoints lie on the sphere is called a *diameter* of the sphere, and the length of such a segment is called *the diameter* of the sphere.