MAC 1140 Pre-Calculus Algebra **Professor Orr**

Finding the Real Zeros of a Polynomial

Review of Basic Facts:

- A *polynomial function* is one of the general form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n,\ a_{n-1},\ \ldots,\ a_1,\ a_0$ are called the *coefficients*, and n is a non-negative integer called the *degree* of the polynomial. We will write $\deg(f) = n$ to indicate that f is of degree n.

- Note that a *constant* polynomial, of the form $f(x) = a_0$, is said to have degree *zero*.

- A number c is called a *zero* of f if $f(c) = 0$. In other words, the zeros of f are the solutions of the equation $f(x) = 0$.

  (For example, the zeros of $f(x) = x^2 - x - 6$ are -2 and 3, because $f(-2) = 0$ and $f(3) = 0$.)

- Given any polynomial equation $g(x) = h(x)$, where g and h are both polynomials, we can always transpose terms to write $g(x) - h(x) = 0$, where $g(x) - h(x)$ will simply be a new polynomial. If we call this new polynomial $f(x)$, then we see that every polynomial equation reduces to the simpler equation $f(x) = 0$. In other words, if we learn how to *find the zeros* of a polynomial, then we will know how to solve any polynomial equation.

Theorems to Know and Use:

1. The Division Algorithm (a special case)

Let $f(x)$ be any polynomial, with $\deg(f) = n$. We can always *divide* f by the monic, linear polynomial $(x - c)$, to obtain a quotient $q(x)$ and a remainder R. As we know, the result can be expressed as follows:

$$\frac{f(x)}{(x-c)} = q(x) + \frac{R}{(x-c)},$$

but it's more useful to write the equivalent form

$$\boldsymbol{f(x) = (x - c)\, q(x) + R.} \qquad (1)$$

Note that we will always have $\deg(q) = n - 1$ and R will be a constant.

Example: If $f(x) = x^4 - x^3 + 2x^2 - 3x + 5$, let's divide f by $(x-2)$.

$$\begin{array}{r|rrrrr} 2 & 1 & -1 & 2 & -3 & 5 \\ & & 2 & 2 & 8 & 10 \\ \hline & 1 & 1 & 4 & 5 & \lfloor 15 \end{array}$$

the quotient is $q(x) = x^3 + x^2 + 4x + 5$ and the remainder is 15. Thus,

$$x^4 - x^3 + 2x^2 - 3x + 5 = (x-2)\left(x^3 + x^2 + 4x + 5\right) + 15. \qquad (2)$$

See that the degree of the quotient q is one less than the degree of f, and the remainder is a constant, as guaranteed by the Division Algorithm.

2. The Remainder Theorem

Since equation (1) above is true for all values of x, it holds in particular when we replace x by c to obtain

$$f(c) = (c-c)\,q(c) + R$$

$$\Rightarrow f(c) = 0 + R$$

$$\Rightarrow f(c) = R. \qquad (3)$$

In other words:

When $f(x)$ is divided by $(x-c)$, the remainder is equal to $f(c)$.

This fact, known as the Remainder Theorem, can be useful in evaluating polynomials, because the operations involved in synthetic division are usually carried out faster and more easily than those involved in directly evaluating the function.

Example: Looking back at the case of $f(x) = x^4 - x^3 + 2x^2 - 3x + 5$, if we had been asked to evaluate $f(2)$, we could have done so by *dividing* f by $(x-2)$ and taking the remainder 15 as the desired result. This is especially easy to see if you look at equation (2) and mentally replace all of the x's by 2's. The left side of the equation will be $f(2)$, and the right side will clearly be equal to 15. Please note that it was much easier to perform the synthetic division than it would be to carry out all of the operations involved in actually evaluating $f(2)$ directly.

3. <u>The Factor Theorem</u>

What's especially useful about the Remainder Theorem is that it allows us to rewrite equation (1) above in the form

$$f(x) = (x - c)\,q(x) + f(c). \qquad (4)$$

This makes it clear that in the very special case where $f(c) = 0$, we have

$$f(x) = (x - c)\,q(x).$$

In other words, if c is a *zero* of f, then $(x - c)$ is a *factor* of f.

On the other hand, if we're given that $(x - c)$ is a factor of f, then we know there is a polynomial $q(x)$ such that $f(x) = (x - c)\,q(x)$, whence a quick look at equation (3) tells us that $f(c) = 0$. So the connection between the *zeros* of f and the *factors* of f goes both ways.

That is,

The number c is a zero of f if, and only if, $(x - c)$ is a factor of f.

The above statement is referred to as the Factor Theorem. It establishes the central fact we need to know about factoring polynomials: that we can arrive at the factored form of the polynomial f if we can find the *zeros* of f.

Example: Let's determine whether $(x + 3)$ is a factor of $f(x) = 2x^6 - 18x^4 + x^2 - 9$. There are two ways we could proceed. One way is to directly evaluate $f(-3)$ to find out whether $f(-3) = 0$. (Be sure you see why we're taking $c = -3$.) But this would be very tedious, even with a calculator handy. The other way is to synthetically divide f by $(x + 3)$ and see if we get a *zero remainder*. (We're using the Remainder Theorem.)

-3	2	0	-18	0	1	0	-9
		-6	18	0	0	-3	9
	2	-6	0	0	1	-3	0 ← Bingo!

Since we got a zero remainder, we conclude that -3 is a zero of f, which in turn (by the Factor Theorem) means that $(x + 3)$ is a factor of f. Moreover, we can use the result of the above division to write f in the following partially factored form:

$$2x^6 - 18x^4 + x^2 - 9 = (x + 3)\,(2x^5 - 6x^4 + x - 3).$$

Be sure you see that the second factor here is simply the *quotient* from the division. We would proceed to factor that fifth-degree quotient even further, except that we have absolutely no idea yet how to even guess what its zeros might be. We need more theorems.

4. Maximum Number of Zeros

In light of the Factor Theorem, it is now easy to see that:

A polynomial of degree n can have at most n zeros.

The reason is simple. Suppose $\deg(f) = n$, and now assume for a moment that f had *more* than n zeros. Since for each zero there is a corresponding linear factor of the form $(x - c)$, the product of all those factors would clearly produce a polynomial of degree greater than n, which contradicts our original assumption. Therefore f can have no more zeros than its degree.

5. Descartes' Rule of Signs

Having established *how many* zeros to expect, the next thing that would be helpful is to know the following: of those zeros that are real numbers, how many are *positive* and how many are *negative*? This question is addressed by a theorem due to Rene Descartes. It has two parts.

(a) Count the number of *variations in sign* of the nonzero coefficients of the polynomial $f(x)$. (The example below will make it clear what this means.) Now the number of *positive* (real) zeros of f is either equal to the number of variations, or else less than that number **by a multiple of two**.

(b) The number of *negative* (real) zeros of f is counted in like manner, except that instead of $f(x)$ we look at $f(-x)$.

Example: $f(x) = x^3 - 2x^2 + 4x - 5$.

Scanning through from left to right, each pair of consecutive coefficients that have *opposite signs* is counted as one variation in sign. The coefficients of $f(x)$ in this case are, taken in order, $+1,\ -2,\ +4,\ -5$. We count three times that the sign changes—that is, 3 variations in sign. We conclude that there may be 3 positive zeros, or else only 1.

Now look at $f(-x) = (-x)^3 - 2(-x)^2 + 4(-x) - 5 = -x^3 - 2x^2 - 4x - 5$. Listing just the coefficients of $f(-x)$, we have $-1,\ -2,\ -4,\ -5$. We count *no* variations in sign. Therefore f can have *no negative zeros.*

Incidentally, it is useful to observe the following short-cut for getting the coefficients of $f(-x)$. The effect of replacing x by $-x$ is that the coefficients of the *even* powers remain the same, while the coefficients of the *odd* powers get reversed.

Another example: $f(x) = x^4 + 2x^3 + x^2 + 5$.

The coefficients of $f(x)$ are $+1, \ +2, \ +1, \ +5$. No variations, so there can be no positive zeros.

Using the above-described short-cut, the coefficients of $f(-x)$ are $+1, -2, \ +1, \ +5$. There are now 2 variations; hence there may be 2 negative zeros, or else there may be none at all.

6. <u>The Rational Zeros Theorem</u>

The next thing we need is a way to really "get our hands dirty" by jumping in and actually testing specific numbers to see if they are zeros of the given polynomial. The key is to have some way of knowing *which* numbers to try. The following theorem is priceless.

Let $f(x)$ be a polynomial with *integer coefficients*, and suppose the rational number $\frac{p}{q}$ (in reduced form) is a zero of f. Then it must be that p is a factor of the *constant term* of f, while q is a factor of the *leading coefficient* of f.

Example: $f(x) = 3x^3 - x^2 - x + 4$.

The candidates for p are the factors of the constant term, 4. They are: $\pm 1, \ \pm 2, \ \pm 4$.

The candidates for q are the factors of the leading coefficient, 3. They are: ± 1, ± 3.

Finally, form all possible fractions $\frac{p}{q}$: ± 1, ± 2, ± 4, $\pm \frac{1}{3}$, $\pm \frac{2}{3}$, $\pm \frac{4}{3}$.

It's very important to understand that there's *no guarantee* that any of the numbers on that list actually *are* zeros of f. What the theorem says is that *if* f has any rational zeros at all, they must be on that list. If we were asked to actually *find* all of the rational zeros of f, then we would need to begin performing synthetic divisions, trying each number in its turn, until we either found three zeros or had tried all twelve numbers on the list, whichever came first.

Example: $f(x) = 6x^3 + 7x^2 - 11x - 12$. Let's find all of the zeros of f, and then give the factored form of f. We will use all of the information gathered so far to proceed in an orderly fashion.

(1) Since $\deg(f) = 3$, f can have at most 3 zeros.

(2) Descartes' Rule of Signs tells us there is exactly one positive zero and that the number of negative zeros may be 2 or 0. (You should check this for yourself by counting variations.)

(3) The Rational Zeros Theorem produces the following rather gargantuan list: ± 1, ± 2, ± 3, ± 4, ± 6, ± 12, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, $\pm\frac{4}{3}$, $\pm\frac{1}{6}$. (Again, you should check this for yourself to make sure you understand how the theorem works.)

(4) Since we know there is one positive zero for sure, while there may not be any negative ones at all, it makes sense to start by checking some of the positive numbers on our list. But in what order? Based on what we know so far, there's no way to say. We would just have to dive in and hope to get lucky. A bit later on we will find another theorem that will help greatly to narrow down the options, so for now let's just pretend we tried eight or ten possibilities before finally stumbling across the one that works, which is $+\frac{4}{3}$.

$\frac{4}{3}$	6	7	-11	-12	
		8	20	12	
	6	15	9	0	← Bingo!

(5) We may now write $f(x) = (x - \frac{4}{3})\,(6x^2 + 15x + 9)$
$= (x - \frac{4}{3})\,3(2x^2 + 5x + 3)$.

(6) To finish the factoring assignment, we look at the second-degree quotient to find *its* zeros. That is, we consider the equation $2x^2 + 5x + 3 = 0$, which we call the *depressed equation*, because it has degree one less than what we started with. Observe that the discriminant ($b^2 - 4ac$) of $2x^2 + 5x + 3$ is $5^2 - 4(2)(3) = 1$, which is a perfect square. This means the polynomial factors over the integers, which we easily accomplish to obtain $(2x + 3)\,(x + 1)$.

(7) Finally, write the fully factored form of f:

$$f(x) = 3(x - \tfrac{4}{3})\,(2x + 3)\,(x + 1)$$
$$= 6(x - \tfrac{4}{3})\,(x + \tfrac{3}{2})\,(x + 1),$$

from which we easily read the zeros $\frac{4}{3}, -\frac{3}{2},$ and -1.

7. <u>The Intermediate Value Theorem</u>

The previous example illustrates just how important it can be to have some way of narrowing down the possibilities before we begin looking for zeros. The next theorem does just that.

Let $f(x)$ be a polynomial. If $f(a)$ and $f(b)$ have *opposite signs*, then there is at least one zero of f between a and b.

The best way to use this Intermediate Value Theorem is to test *consecutive integers* via synthetic division until we find remainders with opposite signs. We will then know that there must be a zero between those integers. And if there is a possible rational zero lying in that interval (from the Rational Zeros Theorem), then it's a good one to try.

Example: $f(x) = 2x^5 + 5x^4 + 14x^3 + 33x^2 - 36x - 108$. Our goal is to factor f completely, and find all of its zeros.

(1) f has at most 5 zeros, because $\deg(f) = 5$.

(2) Descartes' Rule of Signs dictates exactly one positive zero, and allows that the number of negative zeros may be 4, or 2, or 0.

(3) The Rational Zeros Theorem gives us an outrageous list of 32 possibilities! (We'll write just one $\pm$ sign instead of 16, if it's OK.)

$$\pm\left(1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 27,\ 36,\ 54,\ 108,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2},\ \frac{27}{2}\right).$$

(4) First, note that $f(0) = -108$. (In general, $f(0)$ is always equal to the constant term of the polynomial. Why?) Next, we try 1:

$$\begin{array}{r|rrrrrr} 1 & 2 & 5 & 14 & 33 & -36 & -108 \\ & & 2 & 7 & 21 & 54 & 18 \\ \hline & 2 & 7 & 21 & 54 & 18 & -90 \end{array}$$

Since $f(1) = -90$ has the same sign as $f(0) = -108$, there may not be any zeros between 0 and 1. Better to try the next integer, 2:

$$\begin{array}{r|rrrrrr} 2 & 2 & 5 & 14 & 33 & -36 & -108 \\ & & 4 & 18 & 64 & 194 & 316 \\ \hline & 2 & 9 & 32 & 97 & 158 & 208 \end{array}$$

Aha! We now see that $f(1)$ and $f(2)$ have opposite signs. This means there *must* be a zero between 1 and 2. There is no guarantee that the zero in question is a *rational number*, but it's worth a try. Looking at our list of possibilities up in item (3), there's only one number lying between 1 and 2, namely, $\frac{3}{2}$.

$$\begin{array}{r|rrrrrr} \frac{3}{2} & 2 & 5 & 14 & 33 & -36 & -108 \\ & & 3 & 12 & 39 & 108 & 108 \\ \hline & 2 & 8 & 26 & 72 & 72 & 0 \end{array} \leftarrow \text{Bingo!}$$

So we now have $f(x) = \left(x - \frac{3}{2}\right)\left(2x^4 + 8x^3 + 26x^2 + 72x + 72\right)$.

(5) We now stop looking directly at $f(x)$, focusing instead on the depressed equation $q_1(x) = 2x^4 + 8x^3 + 26x^2 + 72x + 72 = 0$. Keeping in mind what Descartes' Rule said in item (2), we stop looking for positive zeros. (Why?) Let's try some negative possibilities. First, note that it's often easier to test the numbers 1 and -1 by direct evaluation rather than synthetic division. It's not hard to compute $q_1(0) = 72$ and $q_1(-1) = 20$. See that there may not be any negative zeros between -1 and 0. Let's go on to try -2:

$$\begin{array}{r|rrrrr} \underline{-2} & 2 & 8 & 26 & 72 & 72 \\ & & -4 & -8 & -36 & -72 \\ \hline & 2 & 4 & 18 & 36 & \lfloor\underline{0} \end{array} \leftarrow \text{Bingo!}$$

Now we can write $f(x) = \left(x - \frac{3}{2}\right)(x+2)\left(2x^3 + 4x^2 + 18x + 36\right)$.

(6) Switching attention once again to the new depressed equation $q_2(x) = 2x^3 + 4x^2 + 18x + 36 = 0$, we observe that since we've found one negative zero, there must be at least one more. (Do you see why?) It's a good idea to keep in mind that just because a number worked once doesn't mean it can't work again. Any zero may have multiplicity greater than one. So we try -2 one more time:

$$\begin{array}{r|rrrr} \underline{-2} & 2 & 4 & 18 & 36 \\ & & -4 & 0 & -36 \\ \hline & 2 & 0 & 18 & \lfloor\underline{0} \end{array} \leftarrow \text{Bingo!}$$

So -2 is a *double* zero, and we now have

$$\begin{aligned} f(x) &= \left(x - \tfrac{3}{2}\right)(x+2)^2\left(2x^2 + 18\right). \\ &= 2\left(x - \tfrac{3}{2}\right)(x+2)^2\left(x^2 + 9\right). \end{aligned}$$

(7) Since the discriminant of the last quotient $q_3(x) = x^2 + 9$ is negative, it has no real zeros. Such a polynomial is called an *irreducible quadratic*. We therefore stop looking for real zeros, and we may list the real zeros of f as $\frac{3}{2}$ and -2 (with multiplicity 2).

We now state two results that are very good to be aware of:

Every polynomial (with real coefficients) can be factored in a unique way as a product of linear and/or irreducible quadratic factors.

And here is an easy corollary that often proves handy:

A polynomial (with real coefficients) of odd degree has at least one real zero.