OBJECTIVE

- Solve maximum and minimum problems using calculus.

A Strategy for Solving Maximum-Minimum Problems:

1. Read the problem carefully. If relevant, make a drawing.
2. Make a list of appropriate variables and constants, noting what varies, what stays fixed, and what units are used. Label the measurements on your drawing, if one exists.
2.5 Maximum-Minimum Problems; Business and Economics Applications

A Strategy for Solving Maximum-Minimum Problems (concluded):

3. Translate the problem to an equation involving a quantity \( Q \) to be maximized or minimized. Try to represent \( Q \) in terms of the variables of step (2).

4. Try to express \( Q \) as a function of one variable. Use the procedures developed in sections 2.1 – 2.3 to determine the maximum or minimum values and the points at which they occur.

Example 2: From a thin piece of cardboard 8 in. by 8 in., square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

1st make a drawing in which \( x \) is the length of each square to be cut.

Example 2 (continued):

2nd write an equation for the volume of the box.

\[
V = l \cdot w \cdot h \\
V(x) = (8 - 2x) \cdot (8 - 2x) \cdot x \\
V(x) = (64 - 32x + 4x^2) \cdot x \\
V(x) = 4x^3 - 32x^2 + 64x \\
\]

Note that \( x \) must be between 0 and 4. So, we need to maximize the volume equation on the interval \((0, 4)\).
Example 2 (continued):

\[ V' = 12x^2 - 64x + 64 = 0 \]
\[ 3x^2 - 16x + 16 = 0 \]
\[ (3x - 4)(x - 4) = 0 \]
\[ 3x - 4 = 0 \quad \text{or} \quad x - 4 = 0 \]
\[ x = \frac{4}{3} \quad \text{or} \quad x = 4 \]

\( x = \frac{4}{3} \) is the only critical value in \((0, 4)\). So, we can use the second derivative.

Example 2 (concluded):

\[ V''(x) = 24x - 64 \]
\[ V''\left( \frac{4}{3} \right) = 24\left( \frac{4}{3} \right) - 64 \]
\[ V''\left( \frac{4}{3} \right) = -32 < 0 \]

Thus, the volume is maximized when the square corners are \( \frac{4}{3} \) inches. The maximum volume is

\[ V\left( \frac{4}{3} \right) = 4\left( \frac{4}{3} \right)^3 - 32\left( \frac{4}{3} \right)^2 + 64\left( \frac{4}{3} \right) \]
\[ V\left( \frac{4}{3} \right) = \frac{25}{27} \text{ in}^3 \]

Example 4:

A stereo manufacturer determines that in order to sell \( x \) units of a new stereo, the price per unit, in dollars, must be \( p(x) = 1000 - x \). The manufacturer also determines that the total cost of producing \( x \) units is given by \( C(x) = 3000 + 2x \).

a) Find the total revenue \( R(x) \).

b) Find the total profit \( P(x) \).

c) How many units must the company produce and sell in order to maximize profit?

d) What is the maximum profit?

e) What price per unit must be charged in order to make this maximum profit?
Example 4 (continued):

a) Revenue = quantity \cdot price
\[ R(x) = x \cdot p \]
\[ R(x) = x(1000 - x) \]
\[ R(x) = 1000x - x^2 \]

b) Profit = Total Revenue – Total Cost
\[ P(x) = R(x) - C(x) \]
\[ P(x) = 1000x - x^2 - (3000 + 20x) \]
\[ P(x) = -x^2 + 980x - 3000 \]

Since there is only one critical value, we can use the second derivative to determine whether or not it yields a maximum or minimum.
\[ P''(x) = -2 \]
Since \( P''(x) \) is negative, \( x = 490 \) yields a maximum. Thus, profit is maximized when 490 units are bought and sold.

Example 4 (concluded):

d) The maximum profit is given by
\[ P(490) = -(490)^2 + 980(490) - 3000 \]
\[ P(490) = 237,100. \]
Thus, the stereo manufacturer makes a maximum profit of $237,100 when 490 units are bought and sold.

e) The price per unit to achieve this maximum profit is
\[ p(490) = 1000 - 490 \]
\[ p(490) = 510. \]
THEOREM 10

Maximum profit occurs at those $x$-values for which

$$R'(x) = C'(x) \quad \text{and} \quad R''(x) < C''(x).$$

Example 5: Promoters of international fund-raising concerts must walk a fine line between profit and loss, especially when determining the price to charge for admission to closed-circuit TV showings in local theaters. By keeping records, a theater determines that, at an admission price of $26, it averages 1000 people in attendance. For every drop in price of $1, it gains 50 customers. Each customer spends an average of $4 on concessions. What admission price should the theater charge in order to maximize total revenue?

Let $x$ be the number of dollars by which the price of $26 should be decreased (if $x$ is negative, the price should be increased).

Revenue = Rev. from tickets + Rev. from concessions

$$R(x) = \# \text{ of people} \cdot \text{ticket price} + \# \text{ of people} \cdot 4$$

$$R(x) = (1000 + 50x)(26 - x) + (1000 + 50x) \cdot 4$$

$$R(x) = 26000 - 1000x + 13000x - 50x^2 + 4000 + 200x$$

$$R(x) = -50x^2 + 500x + 30000$$
Example 5 (continued):
To maximize $R(x)$, we find $R'(x)$ and solve for critical values.

$$R'(x) = -100x + 500 = 0$$
$$-100x = -500$$
$$x = 5$$

Since there is only one critical value, we can use the second derivative to determine if it yields a maximum or minimum.

Example 5 (concluded):
Thus, $x = 5$ yields a maximum revenue. So, the theater should charge

$$26 - 5 = 21$$
per ticket.