SECTION 2.3 Basics of Functions

Objectives

1. Find the domain and range of a relation.
2. Determine whether a relation is a function.
3. Determine whether an equation represents a function.
4. Evaluate a function.
5. Find and simplify a function’s difference quotient.
6. Understand and use piecewise functions.
7. Find the domain of a function.

The answer: See the above list. The question: Who are Celebrity Jeopardy’s five all-time highest earners? The list indicates a correspondence between the five all-time highest earners and their winnings. We can write this correspondence using a set of ordered pairs:

{(Orbach, $34,000), (Shaugnessy, $31,800), (Richter, $29,400), (Schwarzkopf, $28,000), (Stewart, $28,000)}. 
Find the domain and range of a relation.

The mathematical term for a set of ordered pairs is a relation.

**Definition of a Relation**

A relation is any set of ordered pairs. The set of all first components of the ordered pairs is called the **domain** of the relation, and the set of all second components is called the **range** of the relation.

**Example 1: Finding the Domain and Range of a Relation**

Find the domain and range of the relation:

\[
\{(\text{Orbach, }$34,000), (\text{Shaugnessy, }$31,800), (\text{Richter, }$29,400), \\
(\text{Schwarzkopf, }$28,000), (\text{Stewart, }$28,000)\}.
\]

**Solution**

The domain is the set of all first components. Thus, the domain is \{Orbach, Shaugnessy, Richter, Schwarzkopf, Stewart\}.

The range is the set of all second components. Thus, the range is \{$34,000, 31,800, 29,400, 28,000\}.

**Check Point 1**

Find the domain and the range of the relation:

\[
\{(5, 12.8), (10, 16.2), (15, 18.9), (20, 20.7), (25, 21.8)\}.
\]

As you worked Check Point 1, did you wonder if there was a rule that assigned the “inputs” in the domain to the “outputs” in the range? For example, for the ordered pair (15, 18.9), how does the output 18.9 depend on the input 15? Think paid vacation days! The first number in each ordered pair is the number of years a full-time employee has been employed by a medium to large U.S. company. The second number is the average number of paid vacation days each year. Consider, for example, the ordered pair (15, 18.9).

The relation in the vacation-days example can be pictured as follows:

A scatter plot, like the one shown in Figure 2.21, is another way to represent the relation.

**Figure 2.21** The graph of a relation showing a correspondence between years with a company and paid vacation days

Source: Bureau of Labor Statistics

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Determine whether a relation is a function.

| Jersey Orbach   | $34,000 |
| Charles Shaugnessy | $31,800 |
| Andy Richter    | $29,400 |
| Norman Schwarzkopf | $28,000 |
| Jon Stewart     | $28,000 |

### Functions

Shown, again, in the margin are Celebrity Jeopardy’s five all-time highest winners and their winnings. We’ve used this information to define two relations. Figure 2.22(a) shows a correspondence between winners and their winnings. Figure 2.22(b) shows a correspondence between winnings and winners.

A relation in which each member of the domain corresponds to exactly one member of the range is a function. Can you see that the relation in Figure 2.22(a) is a function? Each winner in the domain corresponds to exactly one winning amount in the range. If we know the winner, we can be sure of the amount won. Notice that more than one element in the domain can correspond to the same element in the range. (Schwarzkopf and Stewart both won $28,000.)

Is the relation in Figure 2.22(b) a function? Does each member of the domain correspond to precisely one member of the range? This relation is not a function because there is a member of the domain that corresponds to two members of the range:

- ($28,000, Schwarzkopf)        ($28,000, Stewart).

The member of the domain, $28,000, corresponds to both Schwarzkopf and Stewart in the range. If we know the amount won, $28,000, we cannot be sure of the winner. Because a function is a relation in which no two ordered pairs have the same first component and different second components, the ordered pairs ($28,000, Schwarzkopf) and ($28,000, Stewart) are not ordered pairs of a function.

### Definition of a Function

A function is a correspondence from a first set, called the domain, to a second set, called the range, such that each element in the domain corresponds to exactly one element in the range.

Example 2 illustrates that not every correspondence between sets is a function.

### Example 2  Determining Whether a Relation is a Function

Determine whether each relation is a function:

a. \( \{(1, 6), (2, 6), (3, 8), (4, 9)\} \)  

b. \( \{(6, 1), (6, 2), (8, 3), (9, 4)\} \).
 Functions and Graphs

The word “range” can mean many things, from a chain of mountains to a cooking stove. For functions, it means the set of all function values. For graphing utilities, it means the setting used for the viewing rectangle. Try not to confuse these meanings.

**Study Tip**

The word “range” can mean many things, from a chain of mountains to a cooking stove. For functions, it means the set of all function values. For graphing utilities, it means the setting used for the viewing rectangle. Try not to confuse these meanings.

**Determine whether each relation is a function:**

**a.** \{(1, 2), (3, 4), (5, 6), (5, 8)\}

**b.** \{(1, 2), (3, 4), (6, 5), (8, 5)\}

**Functions as Equations**

Functions are usually given in terms of equations rather than as sets of ordered pairs. For example, here is an equation that models paid vacation days each year as a function of years working for a company:

\[ y = -0.016x^2 + 0.93x + 8.5. \]

The variable \(x\) represents years working for a company. The variable \(y\) represents the average number of vacation days each year. The variable \(y\) is a function of the variable \(x\). For each value of \(x\), there is one and only one value of \(y\). The variable \(x\) is called the independent variable because it can be assigned any value from the domain. Thus, \(x\) can be assigned any positive integer representing the number of years working for a company. The variable \(y\) is called the dependent variable because its value depends on \(x\). Paid vacation days depend on years working for a company. The value of the dependent variable, \(y\), is calculated after selecting a value for the independent variable, \(x\).

We have seen that not every set of ordered pairs defines a function. Similarly, not all equations with the variables \(x\) and \(y\) define a function. If an equation is solved for \(y\) and more than one value of \(y\) can be obtained for a given \(x\), then the equation does not define \(y\) as a function of \(x\).
**Example 3** Determining Whether an Equation Represents a Function

Determine whether each equation defines \( y \) as a function of \( x \):

a. \( x^2 + y = 4 \)  
   b. \( x^2 + y^2 = 4 \).

**Solution** Solve each equation for \( y \) in terms of \( x \). If two or more values of \( y \) can be obtained for a given \( x \), the equation is not a function.

a. \( x^2 + y = 4 \)
   \[ x^2 + y - x^2 = 4 - x^2 \]
   \[ y = 4 - x^2 \]
   This is the given equation. 
   Solve for \( y \) by subtracting \( x^2 \) from both sides. 
   Simplify.

From this last equation we can see that for each value of \( x \), there is one and only one value of \( y \). For example, if \( x = 1 \), then \( y = 4 - 1^2 = 3 \). The equation defines \( y \) as a function of \( x \).

b. \( x^2 + y^2 = 4 \)
   \[ x^2 + y^2 - x^2 = 4 - x^2 \]
   \[ y^2 = 4 - x^2 \]
   Isolate \( y^2 \) by subtracting \( x^2 \) from both sides. 
   Simplify.
   \[ y = \pm \sqrt{4 - x^2} \]
   Apply the square root method.

The \( \pm \) in this last equation shows that for certain values of \( x \) (all values between \(-2 \) and \( 2 \)), there are two values of \( y \). For example, if \( x = 1 \), then \( y = \pm \sqrt{4 - 1^2} = \pm \sqrt{3} \). For this reason, the equation does not define \( y \) as a function of \( x \).

**Check Point** Solve each equation for \( y \) and then determine whether the equation defines \( y \) as a function of \( x \):

a. \( 2x + y = 6 \)
   b. \( x^2 + y^2 = 1 \).

**Function Notation**

When an equation represents a function, the function is often named by a letter such as \( f \), \( g \), \( h \), \( F \), \( G \), or \( H \). Any letter can be used to name a function. Suppose that \( f \) names a function. Think of the domain as the set of the function’s inputs and the range as the set of the function’s outputs. As shown in Figure 2.24, the input is represented by \( x \) and the output by \( f(x) \). The special notation \( f(x) \), read “\( f \) of \( x \)” or “\( f \) at \( x \),” represents the value of the function at the number \( x \).

**Study Tip**

The notation \( f(x) \) does not mean “\( f \) times \( x \).” The notation describes the value of the function at \( x \).

Figure 2.24 A function as a machine with inputs and outputs
Let’s make this clearer by considering a specific example. We know that the equation
\[ y = -0.016x^2 + 0.93x + 8.5 \]
defines \( y \) as a function of \( x \). We'll name the function \( f \). Now, we can apply our new function notation.

Suppose we are interested in finding \( f(10) \), the function’s output when the input is 10. To find the value of the function at 10, we substitute 10 for \( x \). We are evaluating the function at 10.

\[
\begin{align*}
\text{The statement } & f(10) = 16.2, \text{ read “} f \text{ of } 10 \text{ equals } 16.2, \text{“ tells us that the value of the function at 10 is } 16.2. \text{ When the function’s input is } 10, \text{ its output is } 16.2. \text{ (After 10 years, workers average 16.2 vacation days each year.) To find other function values, such as } f(15), f(20), \text{ or } f(23), \text{ substitute the specified input values for } x \text{ into the function’s equation.}
\end{align*}
\]

\[
\begin{align*}
\text{If a function is named } f & \text{ and } x \text{ represents the independent variable, the notation } f(x) \text{ corresponds to the } y\text{-value for a given } x. \text{ Thus,}
\end{align*}
\]

\[
\begin{align*}
f(x) &= -0.016x^2 + 0.93x + 8.5 \quad \text{and} \quad y = -0.016x^2 + 0.93x + 8.5
\end{align*}
\]
define the same function. This function may be written as
\[
y = f(x) = -0.016x^2 + 0.93x + 8.5.
\]

**Example 4** Evaluating a Function

If \( f(x) = x^2 + 3x + 5 \), evaluate:

a. \( f(2) \)  

b. \( f(x + 3) \)  

 c. \( f(-x) \).

**Solution** We substitute 2, \( x + 3 \), and \(-x\) for \( x \) in the definition of \( f \). When replacing \( x \) with a variable or an algebraic expression, you might find it helpful to think of the function’s equation as
\[
f(x) = x^2 + 3x + 5.
\]

a. We find \( f(2) \) by substituting 2 for \( x \) in the equation.

\[
f(2) = 2^2 + 3 \cdot 2 + 5 = 4 + 6 + 5 = 15
\]

Thus, \( f(2) = 15 \).

b. We find \( f(x + 3) \) by substituting \( x + 3 \) for \( x \) in the equation.

\[
f(x + 3) = (x + 3)^2 + 3(x + 3) + 5
\]

\[
= x^2 + 6x + 9 + 3x + 9 + 5
\]

\[
= x^2 + 9x + 23
\]

\[
= (x + 3)^2 + 3(x + 3) + 5
\]
Equivallently,

\[ f(x + 3) = (x + 3)^2 + 3(x + 3) + 5 \]

Square \( x + 3 \) using

\[(A + B)^2 = A^2 + 2AB + B^2.\]

Distribute 3 throughout the parentheses.

Combine like terms.

\[ = x^2 + 6x + 9 + 3x + 9 + 5 \]

\[ = x^2 + 9x + 23. \]

c. We find \( f(-x) \) by substituting \(-x\) for \( x \) in the equation.

Equivallently,

\[ f(-x) = (-x)^2 + 3(-x) + 5 \]

\[ = x^2 - 3x + 5. \]

If \( f(x) = x^2 - 2x + 7 \), evaluate:

a. \( f(-5) \)  

b. \( f(x + 4) \)  

c. \( f(-x) \)

Functions and Difference Quotients

We have seen how slope can be interpreted as a rate of change. In the next section, we will be studying the average rate of change of a function. A ratio, called the difference quotient, plays an important role in understanding the rate at which functions change.

Definition of a Difference Quotient

The expression

\[ \frac{f(x + h) - f(x)}{h} \]

for \( h \neq 0 \) is called the difference quotient.

Example 5 Evaluating and Simplifying a Difference Quotient

If \( f(x) = x^2 + 3x + 5 \), find and simplify:

a. \( f(x + h) \)  

b. \( \frac{f(x + h) - f(x)}{h} \), \( h \neq 0 \).

Solution

a. We find \( f(x + h) \) by replacing \( x \) with \( x + h \) each time that \( x \) appears in the equation.

\[ f(x) = x^2 + 3x + 5 \]

Replace \( x \) with \( x + h \). Replace \( x \) with \( x + h \). Replace \( x \) with \( x + h \). Copy the 5. There is no \( x \) in this term.

\[ f(x + h) = (x + h)^2 + 3(x + h) + 5 \]

\[ = x^2 + 2xh + h^2 + 3x + 3h + 5 \]
Piecewise Functions

The early part of the twentieth century was the golden age of immigration in America. More than 13 million people migrated to the United States between 1900 and 1914. By 1910, foreign-born residents accounted for 15% of the total U.S. population. The graph in Figure 2.25 shows the percentage of Americans who were foreign born throughout the twentieth century.

We can model the data from 1910 through 2000 with two equations, one from 1910 through 1970, years in which the percentage was decreasing, and one from 1970 through 2000, years in which the percentage was increasing. These two trends can be approximated by the function

\[ P(t) = \begin{cases} 
-\frac{11}{60}t + 15 & \text{if } 0 \leq t < 60 \\
\frac{1}{5}t - 8 & \text{if } 60 \leq t \leq 90
\end{cases} \]

in which \( t \) represents the number of years after 1910 and \( P(t) \) is the percentage of foreign-born Americans. A function that is defined by two (or more) equations over a specified domain is called a **piecewise function**.
EXAMPLE 6  Evaluating a Piecewise Function

Use the function \( P(t) \), described previously, to find and interpret:

a. \( P(30) \)  
b. \( P(80) \).

**Solution**

a. To find \( P(30) \), we let \( t = 30 \). Because 30 is less than 60, we use the first line of the piecewise function.

\[
P(t) = -\frac{11}{60}t + 15 \quad \text{This is the function's equation for } 0 \leq t < 60.
\]

\[
P(30) = -\frac{11}{60} \cdot 30 + 15 \quad \text{Replace } t \text{ with } 60.
\]

\[
= 9.5
\]

This means that 30 years after 1910, in 1940, 9.5% of Americans were foreign born.

b. To find \( P(80) \), we let \( t = 80 \). Because 80 is between 60 and 90, we use the second line of the piecewise function.

\[
P(t) = \frac{1}{5}t - 8 \quad \text{This is the function's equation for } 60 \leq t \leq 90.
\]

\[
P(80) = \frac{1}{5} \cdot 80 - 8 \quad \text{Replace } t \text{ with } 80.
\]

\[
= 8
\]

This means that 80 years after 1910, in 1990, 8% of Americans were foreign born.

**Check Point**

If \( f(x) = \begin{cases} x^2 + 3 & \text{if } x < 0 \\ 5x + 3 & \text{if } x \geq 0 \end{cases} \), find:

a. \( f(-5) \)  
b. \( f(6) \).

**The Domain of a Function**

Let's reconsider the function that models the percentage of foreign-born Americans \( t \) years after 1910, up through and including 2000. The domain of this function is

\[
\{0, 1, 2, 3, \ldots, 90\}.
\]

Functions that model data often have their domains explicitly given along with the function’s equation. However, for most functions, only an equation is given, and the domain is not specified. In cases like this, the domain of \( f \) is the largest set of real numbers for which the value of \( f(x) \) is a real number. For example, consider the function

\[
f(x) = \frac{1}{x - 3}.
\]

Because division by 0 is undefined (and not a real number), the denominator \( x - 3 \) cannot be 0. Thus, \( x \) cannot equal 3. The domain of the function consists of all real numbers other than 3, represented by \( \{x \mid x \neq 3\} \). We say that \( f \) is not defined at 3, or \( f(3) \) does not exist.
Just as the domain of a function must exclude real numbers that cause division by zero, it must also exclude real numbers that result in an even root of a negative number. For example, consider the function
\[ g(x) = \sqrt{x}. \]
The equation tells us to take the square root of \( x \). Because only nonnegative numbers have real square roots, the expression under the radical sign, \( x \), must be greater than or equal to 0. The domain of \( g \) is \( \{x | x \geq 0\} \), or the interval \( [0, \infty) \).

Finding a Function's Domain
If a function \( f \) does not model data or verbal conditions, its domain is the largest set of real numbers for which the value of \( f(x) \) is a real number. Exclude from a function's domain real numbers that cause division by zero and real numbers that result in an even root of a negative number.

**EXAMPLE 7** Finding the Domain of a Function

Find the domain of each function:

\[ a. \ f(x) = x^2 - 7x \quad b. \ g(x) = \frac{6x}{x^2 - 9} \quad c. \ h(x) = \sqrt{3x + 12}. \]

**Solution**

a. The function \( f(x) = x^2 - 7x \) contains neither division nor an even root. The domain of \( f \) is the set of all real numbers.

b. The function \( g(x) = \frac{6x}{x^2 - 9} \) contains division. Because division by 0 is undefined, we must exclude from the domain values of \( x \) that cause \( x^2 - 9 \) to be 0. Thus, \( x \) cannot equal –3 or 3. The domain of \( g \) is \( \{x | x \neq -3, x \neq 3\} \).

c. The function \( h(x) = \sqrt{3x + 12} \) contains an even root. Because only nonnegative numbers have real square roots, the quantity under the radical sign, \( 3x + 12 \), must be greater than or equal to 0.

\[ 3x + 12 \geq 0 \]
\[ 3x \geq -12 \]
\[ x \geq -4 \]

The domain of \( h \) is \( \{x | x \geq -4\} \), or the interval \( [-4, \infty) \).

**Check Point**

Find the domain of each function:

\[ a. \ f(x) = x^2 + 3x - 17 \quad b. \ g(x) = \frac{5x}{x^2 - 49} \quad c. \ h(x) = \sqrt{9x - 27}. \]