Section 3-5 Derivatives of Products and Quotients

Derivatives of Products

The derivative properties discussed in the preceding section added substantially to our ability to compute and apply derivatives to many practical problems. In this and the next sections we add a few more properties that will increase this ability even further.

Derivatives of Products

In Section 3-4 we found that the derivative of a sum is the sum of the derivatives. Is the derivative of a product the product of the derivatives?

Let \( F(x) = x^2 \), \( S(x) = x^3 \), and \( f(x) = F(x)S(x) = x^5 \). Which of the following is \( f'(x) \)?

- (A) \( F'(x)S'(x) \)
- (B) \( F(x)S'(x) \)
- (C) \( F'(x)S(x) \)
- (D) \( F(x)S'(x) + F'(x)S(x) \)

Comparing the various expressions computed in Explore-Discuss 1, we see that the derivative of a product is not the product of the derivatives, but appears to involve a slightly more complicated form.

Using the definition of the derivative and the two-step process, it can be shown that

The derivative of the product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.
That is,

**THEOREM 1**  
**Product Rule**  
If  
\[ y = f(x) = F(x)S(x) \]  
and if \( F'(x) \) and \( S'(x) \) exist, then  
\[ f'(x) = F(x)S'(x) + S(x)F'(x) \]  
Also,  
\[ y' = FS' + SF' \]

\[ \frac{dy}{dx} = \frac{dS}{dx} + \frac{dF}{dx} \]

**Example 1**  
**Differentiating a Product**  
Use two different methods to find \( f'(x) \) for  
\[ f(x) = 2x^2(3x^4 - 2). \]

**Solution**  
**Method 1.** Use the product rule:  
\[ f'(x) = 2x^2(3x^4 - 2)' + (3x^4 - 2)(2x^2)' \]
\[ = 2x^2(12x^3) + (3x^4 - 2)(4x) \]
\[ = 24x^5 + 12x^4 - 8x \]
\[ = 36x^5 - 8x \]

**Method 2.** Multiply first; then take derivatives:  
\[ f(x) = 2x^2(3x^4 - 2) = 6x^6 - 4x^2 \]
\[ f'(x) = 36x^5 - 8x \]

**Matched Problem 1**  
Use two different methods to find \( f'(x) \) for  
\[ f(x) = 3x^3(2x^2 - 3x + 1). \]

At this point, all the products we encounter can be differentiated by either of the methods illustrated in Example 1. In the next and later sections we will see that there are situations where the product rule must be used. Unless instructed otherwise, you should use the product rule to differentiate all products in this section to gain experience with the use of this important differentiation rule.

**Example 2**  
**Tangent Lines**  
Let  
\[ f(x) = (2x - 9)(x^2 + 6). \]

(A) Find the equation of the line tangent to the graph of \( f(x) \) at \( x = 3 \).

(B) Find the value(s) of \( x \) where the tangent line is horizontal.

**Solution**  
(A) First, find \( f'(x) \):  
\[ f'(x) = (2x - 9)(x^2 + 6)' + (x^2 + 6)(2x - 9)' \]
\[ = (2x - 9)(2x) + (x^2 + 6)(2) \]

Then, find \( f(3) \) and \( f'(3) \):  
\[ f(3) = [2(3) - 9](3^2 + 6) = (-3)(15) = -45 \]
\[ f'(3) = [2(3) - 9]2(3) + (3^2 + 6)(2) = -18 + 30 = 12 \]
Now, find the equation of the tangent line at \( x = 3 \):

\[
\begin{align*}
y - y_1 &= m(x - x_1) \\
y - f(x_1) &= f'(x_1)(x - x_1) \\
y &= 12x - 81
\end{align*}
\]

(B) The tangent line is horizontal at any value of \( x \) such that \( f'(x) = 0 \), so

\[
\begin{align*}
f'(x) &= (2x - 9)2x + (x^2 + 6)2 = 0 \\
6x^2 - 18x + 12 &= 0 \\
x^2 - 3x + 2 &= 0 \\
(x - 1)(x - 2) &= 0
\end{align*}
\]

\( x = 1, 2 \)

The tangent line is horizontal at \( x = 1 \) and at \( x = 2 \).

**Matched Problem 2**

Repeat Example 2 for \( f(x) = (2x + 9)(x^2 - 12) \).

**Insight**

As Example 2 illustrates, the way we write \( f'(x) \) depends on what we want to do with it. If we are interested only in evaluating \( f'(x) \) at specified values of \( x \), the form in part (A) is sufficient. However, if we want to solve \( f'(x) = 0 \), we must multiply and collect like terms, as we did in part (B).

**Derivatives of Quotients**

As is the case with a product, the derivative of a quotient of two functions is not the quotient of the derivatives of the two functions.

**Explore & Discuss 2**

Let \( T(x) = x^3 \), \( B(x) = x^2 \), and

\[
f(x) = \frac{T(x)}{B(x)} = \frac{x^3}{x^2} = x^3
\]

Which of the following is \( f'(x) \)?

\[
\begin{align*}
(A) \quad & \frac{T'(x)}{B'(x)} \\
(B) \quad & \frac{T'(x)B(x) - T(x)B'(x)}{[B(x)]^2} \\
(C) \quad & \frac{T(x)B'(x)}{[B(x)]^2} \\
(D) \quad & \frac{T'(x)B(x)}{[B(x)]^2} - \frac{T(x)B'(x)}{[B(x)]^2}
\end{align*}
\]

The expressions in Explore & Discuss 2 suggest that the derivative of a quotient leads to a more complicated quotient than you might expect.

In general, if \( T(x) \) and \( B(x) \) are any two differentiable functions and

\[
f(x) = \frac{T(x)}{B(x)}
\]

it can be shown that

\[
f'(x) = \frac{B(x)T'(x) - T(x)B'(x)}{[B(x)]^2}
\]
Thus,

The derivative of the quotient of two functions is the bottom function times the derivative of the top function minus the top function times the derivative of the bottom function, all over the bottom function squared.

**THEOREM 2**  Quotient Rule

If

\[ y = f(x) = \frac{T(x)}{B(x)} \]

and if \( T'(x) \) and \( B'(x) \) exist, then

\[ f'(x) = \frac{B(x)T'(x) - T(x)B'(x)}{[B(x)]^2} \]

Also,

\[ y' = \frac{BT' - TB'}{B^2} \]

\[ \frac{dy}{dx} = \frac{B \frac{dT}{dx} - T \frac{dB}{dx}}{B^2} \]

**Example 3**  Differentiating Quotients

(A) If \( f(x) = \frac{x^2}{2x - 1} \), find \( f'(x) \).

(B) If \( y = \frac{t^2 - t}{t^3 + 1} \), find \( y' \).

(C) Find \( \frac{d}{dx} \frac{x^2 - 3}{x^2} \) by using the quotient rule and also by splitting the fraction into two fractions.

**Solution**

(A) \[ f'(x) = \frac{(2x - 1)(x^2) - x^2(2x - 1)}{(2x - 1)^2} = \frac{(2x - 1)(2x) - x^2(2)}{2(2x - 1)^2} = \frac{4x^2 - 2x - 2x^2}{(2x - 1)^2} = \frac{2x^2 - 2x}{(2x - 1)^2} \]

(B) \[ y' = \frac{(t^3 + 1)(t^2 - t)' - (t^2 - t)(t^3 + 1)'}{(t^3 + 1)^2} = \frac{(t^3 + 1)(2t - 1) - (t^2 - t)(3t^2)}{(t^3 + 1)^2} = \frac{2t^4 - t^3 + 2t - 1 - 3t^4 + 3t^3}{(t^3 + 1)^2} = \frac{-t^4 + 2t^3 + 2t - 1}{(t^3 + 1)^2} \]
(C) **Method 1.** Use the quotient rule:

\[
\frac{d}{dx} \frac{x^2 - 3}{x^2} = \frac{x^2 \frac{d}{dx} (x^2 - 3) - (x^2 - 3) \frac{d}{dx} x^2}{(x^2)^2}
\]

\[
= \frac{x^2 (2x) - (x^2 - 3)2x}{x^4}
\]

\[
= \frac{2x^3 - 2x^3 + 6x}{x^4} = \frac{6x}{x^4} = \frac{6}{x^3}
\]

**Method 2.** Split into two fractions:

\[
\frac{x^2 - 3}{x^2} = \frac{x^2}{x^2} - \frac{3}{x^2} = 1 - 3x^{-2}
\]

\[
\frac{d}{dx} (1 - 3x^{-2}) = 0 - 3(-2)x^{-3} = \frac{6}{x^3}
\]

Comparing methods 1 and 2, we see that it often pays to change an expression algebraically before blindly using a differentiation formula.

**Matched Problem 3**

Find

(A) \( f'(x) \) for \( f(x) = \frac{2x}{x^2 + 3} \)

(B) \( y' \) for \( y = \frac{t^3 - 3t}{t^2 - 4} \)

(C) \( \frac{d}{dx} \frac{2 + x^3}{x^3} \) two ways

**Explore & Discuss 3**

Explain why \( \neq \) is used below, and then find the correct derivative.

\[
\frac{d}{dx} \frac{x^3}{x^2 + 3x + 4} \neq \frac{3x^2}{2x + 3}
\]

**Example 4**

**Sales Analysis** The total sales \( S \) (in thousands of games) for a home video game \( t \) months after the game is introduced are given by

\[
S(t) = \frac{-125t^2}{t^2 + 100}
\]

(A) Find \( S'(t) \).

(B) Find \( S(10) \) and \( S'(10) \). Write a brief verbal interpretation of these results.

(C) Use the results from part (B) to estimate the total sales after 11 months.
Solution

(A) 
\[ S'(t) = \frac{(t^2 + 100)(125t^2) - 125t^2(t^2 + 100)}{(t^2 + 100)^2} \]
\[ = \frac{(t^2 + 100)(250t) - 125t^2(2t)}{(t^2 + 100)^2} \]
\[ = \frac{250t^3 + 25000t - 250t^3}{(t^2 + 100)^2} \]
\[ = \frac{25000t}{(t^2 + 100)^2} \]

(B) 
\[ S(10) = \frac{125(10)^2}{10^2 + 100} = 62.5 \quad \text{and} \quad S'(10) = \frac{25000(10)}{(10^2 + 100)^2} = 6.25 \]

The total sales after 10 months are 62,500 games, and sales are increasing at the rate of 6,250 games per month.

(C) The total sales will increase by approximately 6,250 games during the next month. Thus, the estimated total sales after 11 months are 62,500 + 6,250 = 68,750 games.

Matched Problem 4

Refer to Example 4. Suppose that the total sales \( S \) (in thousands of games) \( t \) months after the game is introduced are given by

\[ S(t) = \frac{150t}{t + 3} \]

(A) Find \( S'(t) \).

(B) Find \( S(12) \) and \( S'(12) \). Write a brief verbal interpretation of these results.

(C) Use the results from part (B) to estimate the total sales after 13 months.

Answers to Matched Problems

1. \( 30x^4 - 36x^3 + 9x^2 \)
2. \( y = 84x - 297 \)
   (A) \( x = -4 \), \( x = 1 \)
3. \( \frac{1}{x+3} - \frac{2x}{(x+3)^2} = \frac{6 - 2x^2}{(x+3)^2} \)
   \( \frac{(t^2 - 4)(3t^2 - 3) - (t^3 - 3t)(2t)}{(t^2 - 4)^2} = \frac{t^4 - 9t^2 + 12}{(t^2 - 4)^2} \)
   (C) \( \frac{6}{x^2} \)
4. \( S'(t) = \frac{450}{(t + 3)^2} \)
   (B) \( S(12) = 120 \); \( S'(12) = 2 \). After 12 months, the total sales are 120,000 games, and sales are increasing at the rate of 2,000 games per month.
   (C) 122,000 games