The workers in a union are concerned that the rate at which wages are increasing is lagging behind the rate of increase in the company’s profits. An automobile dealer wants to predict how badly an anticipated increase in interest rates will decrease his rate of sales. An investor is studying the connection between the rate of increase in the Dow Jones Average and the rate of increase in the Gross Domestic Product over the past 50 years.

In each of these situations there are two quantities—wages and profits in the first instance, for example—that are changing with respect to time. We would like to discover the precise relationship between the rates of increase (or decrease) of the two quantities. We will begin our discussion of such related rates by considering some familiar situations in which the two quantities are distances, and the two rates are velocities.

**Example 1**

**Related Rates and Motion**

A 26-foot ladder is placed against a wall (Fig. 1). If the top of the ladder is sliding down the wall at 2 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 10 feet away from the wall?

**Solution**

Many people reason that since the ladder is of constant length, the bottom of the ladder will move away from the wall at the same rate that the top of the ladder is moving down the wall. This is not the case, as we will see.

At any moment in time, let \( x \) be the distance of the bottom of the ladder from the wall, and let \( y \) be the distance of the top of the ladder on the wall (see Fig. 1). Both \( x \) and \( y \) are changing with respect to time and can be thought of...
as functions of time; that is, \( x = x(t) \) and \( y = y(t) \). Furthermore, \( x \) and \( y \) are related by the Pythagorean relationship:

\[
x^2 + y^2 = 26^2 \tag{1}
\]

Differentiating equation (1) implicitly with respect to time \( t \), and using the chain rule where appropriate, we obtain

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \tag{2}
\]

The rates \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \) are related by equation (2); hence, this type of problem is referred to as a related rates problem.

Now our problem is to find \( \frac{dx}{dt} \) when feet, given that \( \frac{dy}{dt} \) is decreasing at a constant rate of 2 feet per second. We have all the quantities we need in equation (2) to solve for \( \frac{dx}{dt} \), except \( y \). When \( y \) can be found using equation (1):

Substitute and into (2); then solve for \( \frac{dx}{dt} \):

\[
\frac{dx}{dt} = \frac{-2(24)(-2)}{2(10)} = 4.8 \text{ feet per second}
\]

Thus, the bottom of the ladder is moving away from the wall at a rate of 4.8 feet per second.

Insight

In the solution to Example 1, we used equation (1) two ways: first to find an equation relating \( \frac{dy}{dt} \) and \( \frac{dx}{dt} \) and second to find the value of \( y \) when \( x = 10 \). These steps must be done in this order. Substituting \( x = 10 \) and then differentiating does not produce any useful results:

\[
\begin{align*}
x^2 + y^2 &= 26^2 \\
100 + y^2 &= 26^2 \\
0 + 2yy' &= 0 \\
y' &= 0
\end{align*}
\]

Substituting 10 for \( x \) has the effect of stopping the ladder. The rate of change of a stationary object is always 0, but that is not the rate of change of the moving ladder.

A gain, a 26-foot ladder is placed against a wall (Fig. 1). If the bottom of the ladder is moving away from the wall at 3 feet per second, at what rate is the top moving down when the top of the ladder is 24 feet up the wall?

\[
\begin{align*}
(A) & \text{For which values of } x \text{ and } y \text{ in Example 1 is } \frac{dx}{dt} \text{ equal to 2 (that is, the same rate at which the ladder is sliding down the wall)?} \\
(B) & \text{When is } \frac{dx}{dt} \text{ greater than 2? Less than 2?}
\end{align*}
\]
DEFINITION  Suggestions for Solving Related Rates Problems

Step 1. Sketch a figure if helpful.

Step 2. Identify all relevant variables, including those whose rates are given and those whose rates are to be found.

Step 3. Express all given rates and rates to be found as derivatives.

Step 4. Find an equation connecting the variables in step 2.

Step 5. Implicitly differentiate the equation found in step 4, using the chain rule where appropriate, and substitute in all given values.

Step 6. Solve for the derivative that will give the unknown rate.

EXAMPLE 2  Related Rates and Motion  Suppose that two motorboats leave from the same point at the same time. If one travels north at 15 miles per hour and the other travels east at 20 miles per hour, how fast will the distance between them be changing after 2 hours?

Solution  

First, draw a picture, as shown in Figure 2.

All variables, \( x, y, \) and \( z \), are changing with time. Hence, they can be thought of as functions of time; \( x = x(t), y = y(t), \) and \( z = z(t), \) given implicitly. It now makes sense to take derivatives of each variable with respect to time. From the Pythagorean theorem,

\[
z^2 = x^2 + y^2 \tag{3}
\]

We also know that

\[
\frac{dx}{dt} = 20 \text{ miles per hour} \quad \text{and} \quad \frac{dy}{dt} = 15 \text{ miles per hour}
\]

We would like to find \( \frac{dz}{dt} \) at the end of 2 hours; that is, when \( x = 40 \) miles and \( y = 30 \) miles. To do this, we differentiate both sides of equation (3) with respect to \( t \) and solve for \( \frac{dz}{dt} \):

\[
2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \tag{4}
\]

We have everything we need except \( z \). When \( x = 40 \) and \( y = 30 \), we find \( z \) from equation (3) to be 50. Substituting the known quantities into equation (4), we obtain

\[
2(50) \frac{dz}{dt} = 2(40)(20) + 2(30)(15)
\]

\[
\frac{dz}{dt} = 25 \text{ miles per hour}
\]

Thus, the boats will be separating at a rate of 25 miles per hour.

Matched Problem 2  Repeat Example 2 for the situation at the end of 3 hours.

EXAMPLE 3  Related Rates and Motion  Suppose a point is moving along the graph of \( x^2 + y^2 = 25 \) (Fig. 3). When the point is at \((-3, 4)\), its \( x \) coordinate is increasing at the rate of 0.4 unit per second. How fast is the \( y \) coordinate changing at that moment?
Solution  
Since both x and y are changing with respect to time, we can think of each as a function of time:

\[ x = x(t) \quad \text{and} \quad y = y(t) \]

but restricted so that

\[ x^2 + y^2 = 25 \quad (5) \]

Our problem is now to find \( \frac{dy}{dt} \), given and 

Implicitly differentiating both sides of equation (5) with respect to \( t \), we have

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0
\]

Divide both sides by 2.

\[
x \frac{dx}{dt} + y \frac{dy}{dt} = 0
\]

Substitute \( x = -3, y = 4 \), and \( \frac{dx}{dt} = 0.4 \), and solve for \( \frac{dy}{dt} \).

\[
(\text{-3})(0.4) + 4 \frac{dy}{dt} = 0
\]

\[
\frac{dy}{dt} = 0.3 \text{ unit per second}
\]

Matched Problem 3  
A point is moving on the graph of \( y^3 = x^2 \). When the point is at \((-8, 4)\), its y coordinate is decreasing at 2 units per second. How fast is the x coordinate changing at that moment?

Example 4  
Related Rates and Business  
Suppose that for a company manufacturing transistor radios, the cost, revenue, and profit equations are given by

\[
\begin{align*}
C &= 5,000 + 2x & \text{Cost equation} \\
R &= 10x - 0.001x^2 & \text{Revenue equation} \\
P &= R - C & \text{Profit equation}
\end{align*}
\]

where the production output in 1 week is \( x \) radios. If production is increasing at the rate of 500 radios per week when production is 2,000 radios, find the rate of increase in

(A) Cost  \hspace{1cm} (B) Revenue  \hspace{1cm} (C) Profit

Solution  
If production \( x \) is a function of time (it must be, since it is changing with respect to time), then \( C, R, \) and \( P \) must also be functions of time. These functions are implicitly (rather than explicitly) given. Letting \( t \) represent time in weeks, we differentiate both sides of each of the preceding three equations with respect to \( t \), and then substitute \( x = 2,000 \) and \( \frac{dx}{dt} = 500 \) to find the desired rates.

(A) \[ C = 5,000 + 2x \quad \text{Think: } C = C(t) \text{ and } x = x(t). \]

\[
\frac{dC}{dt} = \frac{d}{dt}(5,000) + \frac{d}{dt}(2x)
\]

Differentiate both sides with respect to \( t \).

\[
\frac{dC}{dt} = 0 + 2 \frac{dx}{dt} = 2 \frac{dx}{dt}
\]
Cost is increasing at a rate of $1,000 per week.

Since $\frac{dx}{dt} = 500$ when $x = 2,000$,

$$\frac{dC}{dt} = 2(500) = $1,000 per week$$

Cost is increasing at a rate of $1,000 per week.

(B) $R = 10x - 0.001x^2$

$$\frac{dR}{dt} = \frac{d}{dt}(10x) - \frac{d}{dt}0.001x^2$$

$$\frac{dR}{dt} = 10\frac{dx}{dt} - 0.002x \frac{dx}{dt}$$

$$\frac{dR}{dt} = (10 - 0.002x) \frac{dx}{dt}$$

Since $\frac{dx}{dt} = 500$ when $x = 2,000$,

$$\frac{dR}{dt} = (10 - 0.002(2,000))(500) = $3,000 per week$$

Revenue is increasing at a rate of $3,000 per week.

(C) $P = R - C$

$$\frac{dP}{dt} = \frac{dR}{dt} - \frac{dC}{dt}$$

$$= $3,000 - $1,000 \quad \text{Results from parts (A) and (B)}$$

$$= $2,000 per week$$

Profit is increasing at a rate of $2,000 per week.

Repeat Example 4 for a production level of 6,000 radios per week.

(A) In Example 4 suppose that $x(t) = 500t + 500$. Find the time and production level at which the profit is maximized.

(B) Suppose that $x(t) = t^2 + 492t + 16$. Find the time and production level at which the profit is maximized.

(C) Explain why it is unnecessary to know a formula for $x(t)$ in order to determine the production level at which the profit is maximized.

**Answers to Matched Problems**

1. $\frac{dy}{dt} = -1.25 \text{ ft/sec}$
2. $\frac{dz}{dt} = 25 \text{ mi/hr}$
3. $\frac{dx}{dt} = 6 \text{ units/sec}$
4. (A) $\frac{dC}{dt} = $1,000/wk
   (B) $\frac{dR}{dt} = -$1,000/wk
   (C) $\frac{dP}{dt} = -$2,000/wk