Solving Linear Inequalities in One Variable

A linear inequality in one variable is an inequality which can be put into the form

\[ ax + b > c \]

where \( a, b, \) and \( c \) are real numbers.

Note that the “>” can be replaced by \( \geq, <, \) or \( \leq \).

Examples: Linear inequalities in one variable.

\[ 2x + 3 > 4 \]
\[ 2x - 2 < 6x - 5 \] can be written \( -4x + (-2) < -5 \).
\[ 6x + 1 \geq 3(x - 5) \] can be written \( 6x + 1 \geq -15 \).

The solution set for an inequality can be expressed in two ways.

Example: Express the solution set of \( x < -3 \) in two ways.

1. Set-builder notation: \( \{ x \mid x < -3 \} \)
2. Graph on the real line:

Example: Express the solution set of \( x \leq 4 \) in two ways.

1. Set-builder notation: \( \{ x \mid x \leq 4 \} \)
2. Graph on the real line:
A solution of an inequality in one variable is a number which, when substituted for the variable, results in a true inequality.

Examples: Are any of the values of \(x\) given below solutions of \(2x > 5\)?

\[
\begin{align*}
2 & \quad 2(2) > 5 \quad 4 > 5 \quad \text{False} \\
2.6 & \quad 2(2.6) > 5 \quad 5.2 > 5 \quad \text{True} \\
3 & \quad 2(3) > 5 \quad 6 > 5 \quad \text{True} \\
1.5 & \quad 2(1.5) > 5 \quad 3 > 5 \quad \text{False}
\end{align*}
\]

The solution set of an inequality is the set of all solutions.

Addition and Subtraction Properties

• If \(a > b\) and \(c\) is a real number, then \(a + c > b + c\) and \(a - c > b - c\) have the same solution set.

Example: Solve \(x - 4 > 7\).

\[
x - 4 + 4 > 7 + 4 \quad \text{Add 4 to each side of the inequality.}
\]

\[
x > 11 \quad \{x | x > 11\} \quad \text{Set-builder notation.}
\]

Example: Solve \(3x \leq 2x + 5\).

\[
3x - 2x \leq 2x + 5 - 2x \quad \text{Subtract } 2x \text{ from each side.}
\]

\[
x \leq 5 \quad \{x | x \leq 5\} \quad \text{Set-builder notation.}
\]

Multiplication and Division Properties

• If \(c > 0\) the inequalities \(a > b\), \(ac > bc\), and \(\frac{a}{c} > \frac{b}{c}\) have the same solution set.

• If \(c < 0\) the inequalities \(a > b\), \(ac < bc\), and \(\frac{a}{c} < \frac{b}{c}\) have the same solution set.

Example: Solve \(4x \leq 12\).

\[
\frac{4x}{4} \leq \frac{12}{4} \quad \text{Divide by } 4.
\]

\[
x \leq 3 \quad 4 \text{ is greater than } 0, \text{ so the inequality sign remains the same.}
\]

Example: Solve \(\frac{1}{3}x > 4\).

\[
\frac{1}{3} \cdot (\frac{1}{-3}) < \frac{4}{-3} \quad \text{Multiply by } -3.
\]

\[
x < -12 \quad -3 \text{ is less than } 0, \text{ so the inequality sign changes.}
\]
A compound inequality is formed by joining two inequalities with “and” or “or.”

Example: Solve \( x + 2 < 5 \) and \( 2x - 6 > -8 \).

Solve the first inequality.
\[
x + 2 < 5
\]
\[
\begin{align*}
x &< 3 \quad \text{Solution set} \\
\{x \mid x < 3\}
\end{align*}
\]

Solve the second inequality.
\[
2x - 6 > -8
\]
\[
\begin{align*}
\frac{2x}{2} &> \frac{-8}{2} \quad \text{Add 6} \\
x &> -1 \quad \text{Solution set} \\
\{x \mid x > -1\}
\end{align*}
\]

The solution set of the “and” compound inequality is the intersection of the two solution sets.
\[
\{x \mid x < 3\} \cap \{x \mid x > -1\} = \{x \mid -1 < x < 3\}
\]

When solving compound inequalities, it is possible to work with both inequalities at once.

Example: Solve \( 11 < 6x + 5 < 29 \).

This inequality means \( 11 < 6x + 5 \) and \( 6x + 5 < 29 \).

\[
\begin{align*}
6x + 5 &< 29 \\
11 &< 6x \\
1 &< x \\
\{x \mid 1 < x \}
\end{align*}
\]

Solution set.

Example: Solve \( 8 \geq -\frac{1}{2}x + 6 \geq 5 \).

\[
\begin{align*}
2 &> -\frac{1}{2}x \\
-2 &< x \\
-4 &\leq x \leq 2
\end{align*}
\]

Subtract each part. Multiply each part by \(-2\). Multiplication by a negative number changes the inequality sign for each part. Solution set.
Example: Solve $x + 5 > 6$ or $2x < -4$.

Solve the first inequality. Solve the second inequality.

$x + 5 > 6$ \hspace{1cm} $2x < -4$

$x > 1$ \hspace{1cm} $x < -2$

\{ $x | x > 1$ \} Solution set \hspace{1cm} \{ $x | x < -2$ \} Solution set

Since the inequalities are joined by “or” the solution set is the union of the solution sets.

\{ $x | x > 1$\} \cup \{ $x | x < -2$\}

Example: A cell phone company offers its customers a rate of $89 per month for 350 minutes, or a rate of $40 per month plus $0.50 for each minute used.

How many minutes per month can a customer who chooses the second plan use before the charges exceed those of the first plan?

Let $x$ = the number of minutes used.

Solve the inequality $0.50x + 40 \leq 89$

$0.50x \leq 49$ \hspace{1cm} Subtract 40

$x \leq 98$ \hspace{1cm} Divide by 0.5

The customer can use up to 98 minutes per month before the cost of the second plan exceeds the cost of the first plan.