Radicals and Complex Numbers

Definition of an \( n \)th Root

Radicals

[Diagram showing radical notation: \( \sqrt[n]{X} \)]

- Index
- Radical
- Radicand

Definition of an \( n \)th root

\[ \sqrt[n]{x} = y \quad \text{if and only if} \quad y^n = x \]
Example

\[ \sqrt{8} = 2 \]
because
\[ 2^3 = 8 \]

\[ \sqrt{27} = 3 \]
because
\[ 3^3 = 27 \]

An Ambiguity

\[ \sqrt{25} = 5 \]
because
\[ 5^2 = 25 \]

but it’s also true that…
\[ (-5)^2 = 25 \]

So why not say
\[ \sqrt{25} = -5 \]

Two Answers?

- Roots with an even index always have both a positive and a negative root
  - Because squaring either a negative or a positive gives the same result

To avoid confusion we define the principal root to be the positive root, so:

\[ \sqrt{25} = 5 \quad \text{(not} -5) \]
The Negative Root

- If we want the negative root we use a minus sign:
  \[ -\sqrt{25} = -5 \]

Negative Radicands

- Do Not Confuse \( -\sqrt{25} \) with \( \sqrt{-25} \)

\( \sqrt{-25} \) is not a real number

Negative Radicands

- You cannot take an even root of a negative number because you cannot square any number and get a negative result.

- You can take odd roots of negative numbers:
  \[ \sqrt[3]{-8} = -2 \text{ because } (-2)(-2)(-2) = -8 \]
Some Square Root Identities

\( (\sqrt{x})^2 = x \quad \text{for all non-negative } x \)

\( \sqrt{x^2} = x \quad \text{for all non-negative } x \)

\( \sqrt{x^2} = |x| \quad \text{for all } x \)

A Common Error

\( \sqrt{a+b} \neq \sqrt{a} + \sqrt{b} \)

- for example, you cannot say

\( \sqrt{3^2 + 4^2} = 7 \)  (WRONG!)

- What is the correct result?

First Evaluate Inside

\[
\sqrt{3^2 + 4^2} \\
= \sqrt{9 + 16} \\
= \sqrt{25} \\
= 5
\]
Products

- You can “split up” a radical when it contains a product (not a sum!):

\[ \sqrt{ab} = \sqrt{a} \sqrt{b} \]

- (as long as \( a \) and \( b \) are non-negative)

Example

\[ \sqrt{400} = \sqrt{16 \times 25} \]
\[ = \sqrt{16} \sqrt{25} \]
\[ = 4 \times 5 = 20 \]

Perfect Squares

- Perfect squares are numbers that have whole number square roots: 4, 9, 16, 25, 36, 49, 64, etc.
- All other numbers have irrational roots
Examples

\[ \sqrt{4} = 2 \quad \sqrt{0.04} = 0.2 \]
\[ \frac{4}{9} = \frac{2}{3} \quad \sqrt[3]{64} = 4 \]
\[ \sqrt{-144} = \text{Not a Real Number} \quad \sqrt{-32} = -2 \]

Roots of Variable Expressions

If \( n \) is positive odd integer, then

\[ n\sqrt{a^n} = a. \]

If \( n \) is a positive even integer, then

\[ n\sqrt{a^n} = |a|. \]

Examples

\[ \sqrt{(-3)^2} \quad \text{Index is positive and even} \quad |\text{-3}| = 3 \]
\[ \sqrt[3]{-3^3} \quad \text{Index is positive odd} \quad -3 \]
\[ \sqrt[3]{(x + 2)^3} \quad \text{Index is positive even} \quad |x + 2| \]
\[ \sqrt[3]{(a + b)^3} \quad \text{Index is positive odd} \quad a + b \]