Complex Number System

Definition of the Imaginary Number $i$

The symbol $i$ represents an imaginary number with the properties:

$$i = \sqrt{-1}$$

and

$$i^2 = -1.$$
Definition of $\sqrt{-n}$

For any positive real number $n$, 

$$\sqrt{-n} = \sqrt{-1} \sqrt{n} = i \sqrt{n}.$$ 

Simplifying Expressions in Terms of $i$

- $\sqrt{-64} = 8i$
- $\sqrt{-100} = 10i$
- $\sqrt{-29} = i \sqrt{29}$
- $\sqrt{-150} = i \sqrt{150} = i \sqrt{25 \cdot 6} = 5i \sqrt{6}$

Write the Radical Expression in Terms of $i$

$$\frac{\sqrt{-100}}{\sqrt{-25}} = \frac{10i}{5i} = 2$$
The Complex Number System

Powers of $i$:

- $i = \sqrt{-1}$
- $i^2 = -1$
- $i^3 = i^2 \cdot i = (-1) \cdot i = -i$
- $i^4 = i^2 \cdot i^2 = 1 \cdot 1 = 1$
- $i^5 = i^4 \cdot i = 1 \cdot i = i$

etc...it goes in a cycle.

$i^{273} = (i^{68})^{4} + 1 = 1^{4} \cdot i = 1 \cdot i = i$

Simplifying Powers of $i$

- $i^{13} = (i^4)^{3} + i = i$
- $i^{18} = (i^4)^{4} + i^2 = i^2 = -1$
- $i^{107} = (i^4)^{26} + i^3 = -i$

The Complex Number System

A complex number is an expression of the form $a + bi$

where $a$ and $b$ are real, $i$ is the imaginary unit, and $bi$ denotes multiplication.
The Complex Number System

- In a complex number $a + bi$, either $a$ or $b$ or both can be 0.
- If $b = 0$, as in $5 + 0i = 5$, the number is real. The reals are a subset of the complex numbers.
- If $a = 0$ and $b$ is not 0, as in $0 + 5i = 5i$, the complex number is called pure imaginary.
- If $a$ is not 0 and $b$ is not 0, as in $3 + 2i$, the number is sometimes called imaginary.

The Complex Number System

- The form $a + bi$ is called the Standard Form for the complex number.

Examples:

- $4 + 3i$: 4 is the real part, and 3 is the imaginary part
- $-3 - 5i$: -3 is the real part, and -5 is the imaginary part
- $6$: 6 is the real part, and 0 is the imaginary part
- $0$: 0 is the real part, and 0 is the imaginary part
- $4i$: 0 is the real part, and 4 is the imaginary part

**ADDITION and SUBTRACTION**

\[(a + bi) + (c + di) = (a + c) + (b + d)i\]

\[(2 - 3i) + (4 + 5i) = 6 + 2i\]
\[(5 + 3i) - (8 - 2i) = -3 + 5i\]
\[\sqrt{-36} + \sqrt{-49} = 6i + 7i = 13i\]
MULTIPLICATION

\[(a + bi)(c + di) = ac - bd + (ad + bc)i\]

This is the same as “FOIL”.

Perform the Indicated Operation

\[(5 + 6i)(-5 - 12i) = -25 - 60i - 30i - 72i^2\]
\[= -25 - 90i - 72(1)\]
\[= -25 - 90i - 72\]
\[= -25 - 90i + 72\]
\[= 47 - 90i\]

Determine the Product

\[7i(3 - 4i) = 21i - 28i^2\]
\[21i - 28(-1) = 21i + 28\]
\[= 28 + 21i\]
The Complex Number System

For any complex number $a + bi$, the number $a - bi$ is called its conjugate.

- $5 - 3i$ is the conjugate of $5 + 3i$
- $\frac{2}{3} + 5i$ is the conjugate of $\frac{2}{3} - 5i$
- $-5 - 4i$ is the conjugate of $-5 + 4i$

Let $z = a + bi$. The conjugate of $z$ is denoted $\overline{z} = a - bi$

$3 - 5i = 3 + 5i$

$-\pi - 5.6i = -\pi + 5.6i$

The Complex Number System

The conjugate has the important property that the product of a complex number and its conjugate is always real.

$$(3 - 5i)(3 + 5i) = 3^2 - (5i)^2 = 9 + 25 = 34$$

(More detail:)

$$(3 - 5i)(3 + 5i) = 3 \cdot 3 + 3 \cdot 5i - 5i \cdot 3 - 5i \cdot 5i$$

$= 9 - 25 = 9 + 25 = 34$

The Complex Number System

Division:

- Multiply numerator and denominator by the conjugate of the denominator and simplify.

\[
\frac{2 - 4i}{1 + 3i} \cdot \frac{1 + 3i}{1 + 3i} = \frac{-10 - 10i}{1 + 9} = -1 - i
\]

\[
\frac{6 + 5i}{i} \cdot \frac{i}{i} = \frac{5 - 6i}{1} = 5 - 6i
\]

Check answers: The divisor multiplied by the quotient must equal the dividend.
Find the Product of the Complex Number & Its Conjugate

\[ \frac{5 + i \sqrt{3}}{(5 + i \sqrt{3})(5 - i \sqrt{3})} \]
\[ 5^2 + (\sqrt{3})^2 \]
\[ 25 + 3 = 28 \]

Determine the Quotient

\[ \frac{4}{5i} \]

\[ \text{Rationalize the denominator.} \]
\[ \frac{4 \cdot i}{5i^2} = \frac{4i}{5(-1)} = -\frac{4i}{5} = \frac{4i}{-5} \]

Determine the Quotient

\[ \frac{5 - 2i}{3 - 5i} \]
\[ \left( \frac{5 - 2i}{3 - 5i} \right) \cdot \left( \frac{3 + 5i}{3 + 5i} \right) \]
\[ = \frac{(5 - 2i) \cdot (3 + 5i)}{3^2 + 5^2} \]
Determine the Quotient

\[
\frac{15 + 25i - 6i - 10i^2}{3^2 + 5^2} = \frac{15 + 25i - 6i - 10(-1)}{9 + 25} = \frac{15 + 25i - 6i + 10}{34} = \frac{25 + 19i}{34}
\]

Complex Numbers and Radicals

Be careful!
We have a property that if \(a\) and \(b\) are both positive, \(\sqrt{ab} = \sqrt{a}\sqrt{b}\).

This property is not true if \(a < 0\) and \(b < 0\).
\(\sqrt{(-4)(-9)} = \sqrt{36} = 6\)
But
\(\sqrt{-4}\sqrt{-9} = (2i)(3i) = 6i^2 = -6\)
They are not equal when both numbers are negative.