Symmetry and Transformations of Functions

Definition of Even and Odd Functions

The function \( f \) is an even function if
\[
f(-x) = f(x)
\]
for all \( x \) in the domain of \( f \).
The right side of the equation of an even function does not change if \( x \) is replaced with \(-x\).

The function \( f \) is an odd function if
\[
f(-x) = -f(x)
\]
for all \( x \) in the domain of \( f \).
Every term in the right side of the equation of an odd function changes sign if \( x \) is replaced by \(-x\).

Example

Identify the following function as even, odd, or neither: \( f(x) = 3x^2 - 2 \).

Solution:
We use the given function’s equation to find \( f(-x) \).
\[
f(-x) = 3(-x)^2 - 2 = 3x^2 - 2
\]
The right side of the equation of the given function did not change when we replaced \( x \) with \(-x\).
Because \( f(-x) = f(x) \), \( f \) is an even function.
Even Functions and y-Axis Symmetry

- The graph of an even function in which \( f(-x) = f(x) \) is symmetric with respect to the y-axis.

Odd Functions and Origin Symmetry

- The graph of an odd function in which \( f(-x) = -f(x) \) is symmetric with respect to the origin.

Vertical Shifts

Let \( f \) be a function and \( c \) a positive real number.
- The graph of \( y = f(x) + c \) is the graph of \( y = f(x) \) shifted \( c \) units vertically upward.
- The graph of \( y = f(x) - c \) is the graph of \( y = f(x) \) shifted \( c \) units vertically downward.
Example

• Use the graph of $y = x^2$ to obtain the graph of $y = x^2 + 4$.

Example cont.

• Use the graph of $y = x^2$ to obtain the graph of $y = x^2 + 4$.
   Step 1  Graph $f(x) = x^2$. The graph of the standard quadratic function is shown.
   Step 2  Graph $g(x) = x^2 + 4$. Because we add 4 to each value of $x^2$ in the range, we shift the graph of $f$ vertically 4 units up.

Horizontal Shifts

Let $f$ be a function and $c$ a positive real number:
• The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted to the left $c$ units.
• The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted to the right $c$ units.
Use the graph of \( f(x) \) to obtain the graph of \( h(x) = (x + 1)^2 - 3 \).

**Solution**

Step 1. Graph \( f(x) = x^2 \). The graph of the standard quadratic function is shown.

Step 2. Graph \( g(x) = (x + 1)^2 \). Because we add 1 to each value of \( x \) in the domain, we shift the graph of \( f \) horizontally one unit to the left.

Step 3. Graph \( h(x) = (x + 1)^2 - 3 \). Because we subtract 3, we shift the graph vertically down 3 units.

---

**Reflection about the x-Axis**

- The graph of \( y = -f(x) \) is the graph of \( y = f(x) \) reflected about the x-axis.

---

**Reflection about the y-Axis**

- The graph of \( y = f(-x) \) is the graph of \( y = f(x) \) reflected about the y-axis.
Stretching and Shrinking Graphs

Let $f$ be a function and $c$ a positive real number.

- If $c > 1$, the graph of $y = cf(x)$ is the graph of $y = f(x)$ vertically stretched by multiplying each of its $y$-coordinates by $c$.
- If $0 < c < 1$, the graph of $y = cf(x)$ is the graph of $y = f(x)$ vertically shrunk by multiplying each of its $y$-coordinates by $c$.

Sequence of Transformations

A function involving more than one transformation can be graphed by performing transformations in the following order.

1. Horizontal shifting
2. Vertical stretching or shrinking
3. Reflecting
4. Vertical shifting

Example

- Use the graph of $f(x) = x^3$ to graph $g(x) = (x+3)^3 - 4$

Solution:

Step 1: Because $x$ is replaced with $x+3$, the graph is shifted 3 units to the left.
Example

- Use the graph of \( f(x) = x^3 \) to graph \( g(x) = (x+3)^3 - 4 \)

Solution:

Step 2: Because the equation is not multiplied by a constant, no stretching or shrinking is involved.

Example

- Use the graph of \( f(x) = x^3 \) to graph \( g(x) = (x+3)^3 - 4 \)

Solution:

Step 3: Because \( x \) remains as \( x \), no reflecting is involved.

Example

- Use the graph of \( f(x) = x^3 \) to graph \( g(x) = (x+3)^3 - 4 \)

Solution:

Step 4: Because 4 is subtracted, shift the graph down 4 units.
Symmetry and Transformations of Functions