Polyomial Functions and Their Graphs

Definition of a Polynomial Function

Let $n$ be a nonnegative integer and let $a_n$, $a_{n-1}$, $a_1$, $a_0$ be real numbers with $a_n \neq 0$. The function defined by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

is called a polynomial function of $x$ of degree $n$. The number $a_n$, the coefficient of the variable to the highest power, is called the leading coefficient.

Smooth, Continuous Graphs

Two important features of the graphs of polynomial functions are that they are smooth and continuous. By smooth, we mean that the graph contains only rounded curves with no sharp corners. By continuous, we mean that the graph has no breaks and can be drawn without lifting your pencil from the rectangular coordinate system. These ideas are illustrated in the figure.
The Leading Coefficient Test

As \( x \) increases or decreases without bound, the graph of the polynomial function

\[
f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \quad (a_n \neq 0)
\]

eventually rises or falls. In particular,

1. For \( n \) odd:
   - If the leading coefficient is positive, the graph falls to the left and rises to the right.
   - If the leading coefficient is negative, the graph rises to the left and falls to the right.

2. For \( n \) even:
   - If the leading coefficient is positive, the graph rises to the left and to the right.
   - If the leading coefficient is negative, the graph falls to the left and to the right.

Text Example

Use the Leading Coefficient Test to determine the end behavior of the graph of the quadratic function \( f(x) = x^3 + 3x^2 - x - 3 \).

Solution: Because the degree is odd (\( n = 3 \)) and the leading coefficient, 1, is positive, the graph falls to the left and rises to the right, as shown in the figure.
We now have a polynomial equation.

We find the zeros of \( f \) by setting \( f(x) \) equal to 0.

\[
\begin{align*}
-x^4 + 4x^3 - 4x^2 &= 0 \\
x^4 - 4x^3 + 4x^2 &= 0 \\
x^2(x^2 - 4x + 4) &= 0 \\
x^2(2-x)^2 &= 0 \\
x^2 &= 0 \quad \text{or} \quad (2-x)^2 = 0 \\
x &= 0 \quad \text{or} \quad x = 2
\end{align*}
\]
Set each factor equal to zero.

Solve for \( x \).

Find all zeros of \( f(x) = -x^4 + 4x^3 - 4x^2 \).

\textbf{Solution}

- We find the zeros of \( f \) by setting \( f(x) \) equal to 0.
- We now have a polynomial equation.
- Multiply both sides by \(-1\) (optional step)
- Factor out \( x^2 \).
- Factor completely.
- Set each factor equal to zero.
- Solve for \( x \).

\textbf{Text Example}

\textbf{Multiplicity and x-Intercepts}

If \( r \) is a zero of even multiplicity, then the graph \textbf{touches} the \textbf{x}-axis and turns around at \( r \). If \( r \) is a zero of odd multiplicity, then the graph \textbf{crosses} the \textbf{x}-axis at \( r \). Regardless of whether a zero is even or odd, graphs tend to flatten out at zeros with multiplicity greater than one.

\textbf{Example}

- Find the x-intercepts and multiplicity of \( f(x) = 2(x+2)^2(x-3) \)

\textbf{Solution:}
- \( x = -2 \) is a zero of multiplicity 2 or even
- \( x = 3 \) is a zero of multiplicity 1 or odd
Graphing a Polynomial Function

\[ f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \quad (a_n \neq 0) \]

1. Use the Leading Coefficient Test to determine the graph's end behavior.

2. Find \( x \)-intercepts by setting \( f(x) = 0 \) and solving the resulting polynomial equation. If there is an \( x \)-intercept at \( r \) as a result of \((x - r)^k \) in the complete factorization of \( f(x) \), then:
   a. If \( k \) is even, the graph touches the \( x \)-axis at \( r \) and turns around.
   b. If \( k \) is odd, the graph crosses the \( x \)-axis at \( r \).
   c. If \( k > 1 \), the graph flattens out at \((r, 0)\).

3. Find the \( y \)-intercept by setting \( x \) equal to 0 and computing \( f(0) \).

4. Use symmetry, if applicable, to help draw the graph:
   a. \( y \)-axis symmetry: \( f(-x) = f(x) \)
   b. Origin symmetry: \( f(-x) = -f(x) \).

5. Use the fact that the maximum number of turning points of the graph is \( n - 1 \) to check whether it is drawn correctly.

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Graphing a Polynomial Function

\[ f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \quad (a_n \neq 0) \]

4. Use symmetry, if applicable, to help draw the graph:
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Text Example

Graph: \( f(x) = x^4 - 2x^2 + 1 \).

**Solution**

**Step 1** Determine end behavior. Because the degree is even \((n = 4)\) and the leading coefficient, 1, is positive, the graph rises to the left and the right.
Graph: \( f(x) = x^4 - 2x^2 + 1 \).

**Solution**

**Step 1** Find the \( x \)-intercepts (zeros of the function) by setting \( f(x) = 0 \).

\[
x^4 - 2x^2 + 1 = 0
\]

\[
(x^2 - 1)(x^2 - 1) = 0
\]

Factor.

\[
(x + 1)(x - 1)(x + 1)(x - 1) = 0
\]

Factor completely.

\[
(x + 1)^2(x - 1)^2 = 0
\]

Express the factoring in more compact notation.

\[
(x + 1)^2 = 0 \quad \text{or} \quad (x - 1)^2 = 0
\]

Set each factor equal to zero.

\[
x = -1 \quad \text{or} \quad x = 1
\]

Solve for \( x \).

**Step 2** Find the \( x \)-intercepts (zeros of the function) by setting \( f(x) = 0 \).

\[
(x^2 - 1)(x^2 - 1) = 0
\]

Factor.

\[
(x + 1)(x - 1)(x + 1)(x - 1) = 0
\]

Factor completely.

\[
(x + 1)^2(x - 1)^2 = 0
\]

Express the factoring in more compact notation.

\[
(x + 1)^2 = 0 \quad \text{or} \quad (x - 1)^2 = 0
\]

Set each factor equal to zero.

\[
x = -1 \quad \text{or} \quad x = 1
\]

Solve for \( x \).

We see that -1 and 1 are both repeated zeros with multiplicity 2. Because of the even multiplicity, the graph touches the \( x \)-axis at -1 and 1 and turns around. Furthermore, the graph tends to flatten out at these zeros with multiplicity greater than one:

- \( x \)-intercepts: \( -1 \) and \( 1 \)
- \( y \)-intercept: \( (0, 1) \)

**Step 3** Find the \( y \)-intercept. Replace \( x \) with 0 in \( f(x) = -x^4 + 4x - 1 \).

\[
f(0) = 0^4 - 2(0)^2 + 1 = 1
\]

There is a \( y \)-intercept at 1, so the graph passes through \( (0, 1) \).
Text Example cont.

Graph: \( f(x) = x^4 - 2x^2 + 1 \).

**Solution**

**Step 4** Use possible symmetry to help draw the graph. Our partial graph suggests \( y \)-axis symmetry. Let's verify this by finding \( f(-x) \).

\[
\begin{align*}
  f(x) &= x^4 - 2x^2 + 1 \\
  \text{Replace } x \text{ with } -x \\
  f(-x) &= (-x)^4 - 2(-x)^2 + 1 = x^4 - 2x^2 + 1
\end{align*}
\]

Because \( f(-x) = f(x) \), the graph of \( f \) is symmetric with respect to the \( y \)-axis. The following figure shows the graph.

Text Example cont.

Graph: \( f(x) = x^4 - 2x^2 + 1 \).

**Solution**

**Step 5** Use the fact that the maximum number of turning points of the graph is \( n - 1 \) to check whether it is drawn correctly. Because \( n = 4 \), the maximum number of turning points is \( 4 - 1 \), or 3. Because our graph has three turning points, we have not violated the maximum number possible.

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