Quadratic and Rational Inequalities

Definition of a Quadratic Inequality

A quadratic inequality is any inequality that can be put in one of the forms

\[ ax^2 + bx + c < 0 \]
\[ ax^2 + bx + c > 0 \]
\[ ax^2 + bx + c \leq 0 \]
\[ ax^2 + bx + c \geq 0 \]

where \( a, b, \) and \( c \) are real numbers and \( a \neq 0 \).

Procedure for Solving Quadratic Inequalities

- Express the inequality in the standard form \[ ax^2 + bx + c < 0 \] or \[ ax^2 + bx + c > 0 \].
- Solve the equation \( ax^2 + bx + c = 0 \). The real solutions are the boundary points.
- Locate these boundary points on a number line, thereby dividing the number line into test intervals.
- Choose one representative number within each test interval. If substituting that value into the original inequality produces a true statement, then all real numbers in the test interval belong to the solution set. If substituting that value into the original inequality produces a false statement, then no real numbers in the test interval belong to the solution set.
- Write the solution set; the interval(s) that produced a true statement.
Example
Solve and graph the solution set on a real number line:

\[ 2x^2 - 3x \geq 2. \]

Solution

Step 1  Write the inequality in standard form. We can write by subtracting 2 from both sides to get zero on the right.

\[ 2x^2 - 3x - 2 \geq 0. \]

Step 2  Solve the related quadratic equation. Replace the inequality sign with an equal sign. Thus, we will solve.

\[ 2x^2 - 3x - 2 = 0 \]

This is the related quadratic equation.

\[ (2x + 1)(x - 2) = 0 \]

Factor.

\[ 2x + 1 = 0 \quad \text{or} \quad x - 2 = 0 \]

Set each factor equal to 0.

\[ x = -\frac{1}{2} \quad \text{or} \quad x = 2 \]

Solve for \( x \).

The boundary points are \(-\frac{1}{2}\) and 2.

Solution cont.
Solve and graph the solution set on a real number line:

\[ 2x^2 - 3x \geq 2. \]

Solution

Step 3  Locate the boundary points on a number line. The number line with the boundary points is shown as follows:

The boundary points divide the number line into three test intervals. Including the boundary points (because of the given greater than or equal to sign), the intervals are \( (-\infty, -1/2] \), \( [-1/2, 2] \), \( [2, \infty) \).

Solution cont.
Solve and graph the solution set on a real number line:

\[ 2x^2 - 3x \geq 2. \]

Solution

Step 4  Take one representative number within each test interval and substitute that number into the original inequality.

<table>
<thead>
<tr>
<th>Test Interval</th>
<th>Representative Number</th>
<th>Substitute into ( 2x^2 - 3x \geq 2 )</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (-\infty, -1/2] )</td>
<td>-1</td>
<td>( (2)(-1) - 3(-1) \leq 2 )</td>
<td>( (-\infty, -1/2] ) belongs to the solution set.</td>
</tr>
<tr>
<td>( [-1/2, 2] )</td>
<td>0</td>
<td>( (2)(0) - 3(0) \leq 2 )</td>
<td>( [-1/2, 2] ) does not belong to the solution set.</td>
</tr>
<tr>
<td>( [2, \infty) )</td>
<td>3</td>
<td>( (2)(3) - 3(3) \leq 2 )</td>
<td>( [2, \infty) ) belongs to the solution set.</td>
</tr>
</tbody>
</table>
Example cont.
Solve and graph the solution set on a real number line:
\[ 2x^2 - 3x \geq 2. \]

Solution
Step 5 The solution set are the intervals that produced a true statement. Our analysis shows that the solution set is \((\neg\infty, -1/2]\) or \([2, \infty)\).

The graph of the solution set on a number line is shown as follows:

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Text Example
Solve and graph the solution set: \[ \frac{x + 1}{x + 3} \leq 2 \]

Solution
Step 1 Express the inequality so that one side is zero and the other side is a single quotient. We subtract 2 from both sides to obtain zero on the right.

\[
\begin{align*}
x + 1 &\leq 2 \\
x + 3 &> 0 \\
x + 3 &\geq 0 \\
x + 3 &\geq 0 \\
x - 1 &\leq 0
\end{align*}
\]

These are the values that make the previous quotient zero or undefined.

Step 2 Find boundary points by setting the numerator and the denominator equal to zero.

\[
\begin{align*}
x + 5 &= 0 \\
x + 3 &= 0 \\
x &= -5 \\
x &= -3
\end{align*}
\]

Set the numerator and denominator equal to 0. These are the values that make the previous quotient zero or undefined.

Solve for \(x\).

The boundary points are -5 and -3. Because equality is included in the given less-than-or-equal-to symbol, we include the value of \(x\) that causes the quotient to be zero. Thus, -5 is included in the solution set. By contrast, we do not include 1 in the solution set because -3 makes the denominator zero.

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Step 3 Locate boundary points on a number line.

The boundary points divide the number line into three test intervals, namely \((-\infty, -5], [-5, -3), (-3, \infty)\).

\[
\begin{align*}
x + \frac{1}{x + 3} &\leq 2 \\
\text{Step 4} &\text{ Take one representative number within each test interval and substitute that number into the original equality.} \\
\hline 
\text{Test Interval} & \hline 
\text{Representative Number} & \substitute{\frac{x + 1}{x + 3}} & \leq 2 & \text{Conclusion} \\
\langle -\infty, -5 \rangle & 0 & \substitute{-6 + 1}{-6 + 3} & \leq 2 & \text{True} \\
\langle -5, -3 \rangle & 2 & \substitute{-4 + 1}{-4 + 3} & \leq 2 & \text{False} \\
\langle -3, \infty \rangle & 5 & \substitute{0 + 1}{0 + 3} & \leq 2 & \text{True} \\
\end{align*}
\]

Step 5 The solution set are the intervals that produced a true statement. Our analysis shows that the solution set is \((-\infty, -5]\) or \((-3, \infty)\).
The Position Formula for a Free-Falling Object Near Earth’s Surface

An object that is falling or vertically projected into the air has its height in feet above the ground given by

\[ s = -16t^2 + v_0 t + s_0 \]

where \( s \) is the height in feet, \( v_0 \) is the original velocity (initial velocity) of the object in feet per second, \( t \) is the time that the object is in motion in seconds, and \( s_0 \) is the original height (initial height) of the object in feet.

Example

An object is propelled straight up from ground level with an initial velocity of 80 fps. Its height at time \( t \) is described by \( s = -16t^2 + 80t \) where the height, \( s \), is measured in feet and the time, \( t \), is measured in seconds. In which time interval will the object be more than 64 feet above the ground?

Solution

\[ -16t^2 + 80t > 64 \]
This is the inequality implied by the problem’s question. We must find \( t \).

\[ -16t^2 + 80t - 64 = 0 \]
Subtract 64 from both sides.

\[ -16(t^2 - 5t - 4) = 0 \]
Solve the related quadratic equation.

\[ (t - 1)(t - 4) = 0 \]
Factor.

\[ t - 1 = 0 \text{ or } t - 4 = 0 \]
Set each factor equal to 0.

\[ t = 1 \text{ or } t = 4 \]
Solve for \( t \). The boundary points are 1 and 4.

Example cont.

An object is propelled straight up from ground level with an initial velocity of 80 fps. Its height at time \( t \) is described by \( s = -16t^2 + 80t \) where the height, \( s \), is measured in feet and the time, \( t \), is measured in seconds. In which time interval will the object be more than 64 feet above the ground?

Solution

\[ -16t^2 + 80t > 64 \]
This is the inequality implied by the problem’s question. We must find \( t \).

\[ t = 1 \text{ or } t = 4 \]
The boundary points are 1 and 4.

Since neither boundary point satisfy the inequality, 1 and 4 are not part of the solution.

With test intervals \((-\infty, 1), (1, 4), \text{ and } (4, \infty)\), we could use 0, 2, and 5 as test points for our analysis.
Example cont.

An object is propelled straight up from ground level with an initial velocity of 80 fps. Its height at time \( t \) is described by \( s = -16t^2 + 80t \) where the height, \( s \), is measured in feet and the time, \( t \), is measured in seconds. In which time interval will the object be more than 64 feet above the ground?

**Solution**

<table>
<thead>
<tr>
<th>Test Interval</th>
<th>Representative Number</th>
<th>Substitute into ((x - 1)(x - 4) &lt; 0)</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, 1))</td>
<td>0</td>
<td>(-16(0)^2 + 80(0) &lt; 0)</td>
<td>0 (\neq 0), False</td>
</tr>
<tr>
<td>((1, 4))</td>
<td>2</td>
<td>(-16(2)^2 + 40 &lt; 0)</td>
<td>((1, 4)) belongs to the solution set</td>
</tr>
<tr>
<td>((4, \infty))</td>
<td>5</td>
<td>(-16(5)^2 + 20 &lt; 0)</td>
<td>((4, \infty)) does not belong to the solution set</td>
</tr>
</tbody>
</table>

The object will be above 64 feet between 1 and 4 seconds.

**Quadratic and Rational Inequalities**