The Hyperbola

Definition of a Hyperbola
• A hyperbola is the set of points in a plane the difference whose distances from two fixed points (called foci) is constant.

Standard Forms of the Equations of a Hyperbola
• The standard form of the equation of the hyperbola with center at the origin is
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \] or \[ \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \]
• The left equation has a transverse axis that lies on the x-axis shown in the graph on the left. The right equation’s transverse axis lies on the y-axis. The vertices are \(a\) units from the center and the foci are \(c\) units from the center. For both equations, \(b^2 = c^2 - a^2\).
Find the standard form of the equation of a hyperbola with foci at (0, -3) and (0, 3) and vertices (0, -2) and (0, 2).

**Solution**
Because the foci are located at (0, -3) and (0, 3), on the y-axis, the transverse axis lies on the y-axis. The center of the hyperbola is midway between the foci, located at (0, 0). Thus, the form of the equation is

\[
\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1
\]

We need to determine the values for \(a^2\) and \(b^2\). The distance from the center (0, 0) to either vertex, (0, -2) or (0, 2) is 2, so \(a = 2\).

\[
\frac{y^2}{2^2} - \frac{x^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{4} - \frac{x^2}{b^2} = 1
\]

We must still find \(b^2\). The distance from the center, (0, 0), to either focus, (0, -3) or (0, 3) is 3. Thus, \(c = 3\). Because \(b^2 = c^2 - a^2\), we have \(b^2 = 3^2 - 2^2 = 9 - 4 = 5\).

Substituting 5 for \(b^2\) in the last equation gives us the standard form of the hyperbola’s equation. The equation is

\[
\frac{y^2}{4} - \frac{x^2}{5} = 1
\]

**The Asymptotes of a Hyperbola Centered at the Origin**

- The hyperbola \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\) has a horizontal transverse axis and two asymptotes
  \[y = \pm \frac{b}{a} x\]

- The hyperbola \(\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1\) has a vertical transverse axis and two asymptotes
  \[y = \pm \frac{a}{b} x\]
Graphing Hyperbolas

1. Locate the vertices.
2. Use dashed lines to draw the rectangle centered at the origin with sides parallel to the axes, crossing one axis at \( \pm a \) and the other at \( \pm b \).
3. Use dashed lines to draw the diagonals of this rectangle and extend them to obtain the asymptotes.
4. Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes.

Example

• Graph: \( \frac{(x-3)^2}{4} - \frac{(y-1)^2}{25} = 1 \)

Solution:

• Center (3, 1)
  \( a=2, b=5 \)
  \( c^2 = a^2 + b^2 = 4 + 25 = 29 \)
  \( y = \pm \frac{5}{2}(x-3) + 1 \)

Example cont.
Standard Forms of Equations of Hyperbolas Centered at (h,k)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Center</th>
<th>Transverse Axis</th>
<th>Foci</th>
<th>Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 )</td>
<td>(h, k)</td>
<td>Parallel to the x-axis, horizontal</td>
<td>(h + c, k)</td>
<td>(h - c, k)</td>
</tr>
<tr>
<td>( \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{c^2} = 1 )</td>
<td>(h, k)</td>
<td>Parallel to the x-axis, vertical</td>
<td>(h, k + c)</td>
<td>(h, k - c)</td>
</tr>
<tr>
<td>( \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 )</td>
<td>(h, k)</td>
<td>Parallel to the y-axis, horizontal</td>
<td>(h, k + a)</td>
<td>(h, k - a)</td>
</tr>
<tr>
<td>( \frac{(x-h)^2}{b^2} - \frac{(y-k)^2}{a^2} = 1 )</td>
<td>(h, k)</td>
<td>Parallel to the y-axis, vertical</td>
<td>(h, k + a)</td>
<td>(h, k - a)</td>
</tr>
</tbody>
</table>

Graph: Where are the foci located?

We see that \( h = 2 \) and \( k = 3 \). Thus, the center of the hyperbola, \((h, k)\), is \((2, 3)\).

We can graph the hyperbola by using vertices, asymptotes, and our four-step graphing procedure.

Text Example cont.

Solution: In order to graph the hyperbola, we need to know its center \((h, k)\). In the standard form of the equations centered at \((h, k)\), \(h\) is the number subtracted from \(x\) and \(k\) is the number subtracted from \(y\).

\[
\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1
\]

We see that \( h = 2 \) and \( k = 3 \). Thus, the center of the hyperbola, \((2, 3)\), is \((2, 3)\).

We can graph the hyperbola by using vertices, asymptotes, and our four-step graphing procedure.
Step 2  Draw a rectangle. Because $a^2 = 16$ and $b^2 = 9$, $a = 4$ and $b = 3$. The rectangle passes through points that are 4 units to the right and left of the center (the vertices are located here) and 3 units above and below the center. The rectangle is shown using dashed lines.

Solution

Step 3  Draw extended diagonals of the rectangle to obtain the asymptotes. We draw dashed lines through the opposite corners of the rectangle to obtain the graph of the asymptotes. The equations of the asymptotes of the hyperbola are $y - 3 = \pm \frac{3}{4} (x - 2)$.

Step 4  Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes. The hyperbola is shown below. The foci are located $c$ units to the right and left of the center. We find $c$ using $c^2 = a^2 + b^2$.

$c^2 = 16 + 9 = 25$

Because $c^2 = 25$, $c = 5$. This means that the foci are 5 units to the right and left of the center, $(2, 3)$. Five units to the right of $(2, 3)$ puts one focus at $(7, 3)$. Five units to the left of the center puts the other focus at $(-3, 3)$. 

Solution
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