Factoring
Polynomials

Factoring is the process of writing a polynomial as the product of two or more polynomials. The factors of $6x^2 - x - 2$ are $2x + 1$ and $3x - 2$. In this section, we will be factoring over the integers. Polynomials that cannot be factored using integer coefficients are called irreducible over the integers or prime.

The goal in factoring a polynomial is to use one or more factoring techniques until each of the polynomial’s factors is prime or irreducible. In this situation, the polynomial is said to be factored completely.

Common Factors

In any factoring problem, the first step is to look for the greatest common factor. The greatest common factor is a expression of the highest degree that divides each term of the polynomial. The distributive property in the reverse direction

$ab + ac = a(b + c)$

can be used to factor out the greatest common factor.
A Strategy for Factoring $ax^2 + bx + c$

(Assume, for the moment, that there is no greatest common factor.)

1. Find two First terms whose product is $ax^2$: $\boxed{x \times x = ax^2}$

2. Find two Last terms whose product is $c$: $\boxed{x \times x = c}$

3. By trial and error, perform steps 1 and 2 until the sum of the Outside product and Inside product is $b$:

   $\boxed{\text{sum of O + I}}$ = $b$

If no such combinations exist, the polynomial is prime.

Text Example

Factor: a. $18x^3 + 27x^2$  b. $x^3(x + 3) + 5(x + 3)$

Solution

a. We begin by determining the greatest common factor. 9 is the greatest integer that divides 18 and 27. Furthermore, $x^2$ is the greatest expression that divides $x^3$ and $x^2$. Thus, the greatest common factor of the two terms in the polynomial is $9x^2$.

$$18x^3 + 27x^2 = 9x^2(2x) + 9x^2(3)$$

Express each term with the greatest common factor as a factor.

$$= 9x^2(2x + 3)$$

Factor out the greatest common factor.

b. In this situation, the greatest common factor is the common binomial factor $(x + 3)$. We factor out this common factor as follows.

$$x^3(x + 3) + 5(x + 3) = (x + 3)(x^3 + 5)$$

Factor out the common binomial factor.

Text Example

Factor: a. $x^2 + 6x + 8$  b. $x^2 + 3x - 18$

Solution

a. The factors of the first term are $x$ and $x$ $(x \times x)$

To find the second term of each factor, we must find two numbers whose product is 8 and whose sum is 6.

<table>
<thead>
<tr>
<th>Factors of 8</th>
<th>Sum of Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 8</td>
<td>9</td>
</tr>
<tr>
<td>2, 4</td>
<td>6</td>
</tr>
<tr>
<td>-1, -8</td>
<td>-9</td>
</tr>
<tr>
<td>-2, -4</td>
<td>-6</td>
</tr>
</tbody>
</table>

This is the desired sum.

From the table above, we see that 4 and 2 are the required integers. Thus, $x^2 + 6x + 8 = (x + 4)(x + 2)$ or $(x + 2)(x + 4)$.
Text Example cont.

Factor: a. $x^2 + 6x + 8$  
     b. $x^2 + 3x - 18$

Solution

b. We begin with $x^2 + 3x - 18 = (x \quad )(x \quad )$.
To find the second term of each factor, we must find two numbers whose
product is $-18$ and whose sum is $3$.

<table>
<thead>
<tr>
<th>Factors of $-18$</th>
<th>Sum of Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$18, -1$</td>
<td>$17$</td>
</tr>
<tr>
<td>$18, -1$</td>
<td>$17$</td>
</tr>
<tr>
<td>$9, -2$</td>
<td>$-7$</td>
</tr>
<tr>
<td>$9, -2$</td>
<td>$-7$</td>
</tr>
<tr>
<td>$6, -3$</td>
<td>$3$</td>
</tr>
<tr>
<td>$6, -3$</td>
<td>$3$</td>
</tr>
<tr>
<td>$-6, 3$</td>
<td>$-3$</td>
</tr>
<tr>
<td>$-6, 3$</td>
<td>$-3$</td>
</tr>
</tbody>
</table>

From the table above, we see that $6$ and $-3$ are the required integers. Thus,

$$x^2 + 3x - 18 = (x + 6)(x - 3) \quad \text{or} \quad (x - 3)(x + 6).$$

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Text Example

Factor: $8x^2 - 10x - 3$.

Solution

Step 1  Find two First terms whose product is $8x^2$.

$$8x^2 - 10x - 3 = (8x \quad )(x \quad )$$

Step 2  Find two Last terms whose product is $-3$. The possible factors are $1(-3)$ and $-1(3)$.

Step 3  Try various combinations of these factors. The correct factorization of $8x^2 - 10x - 3$ is the one in which the sum of the Outside and Inside products is equal to $-10x$. Here is a list of possible factors.

<table>
<thead>
<tr>
<th>Possible Factors of $8x^2 - 10x - 3$</th>
<th>Sum of Outside and Inside Products (Should Equal $-10x$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(8x + 10x + 3)$</td>
<td>$-24x - a - 25ax$</td>
</tr>
<tr>
<td>$(8x - 10x - 3)$</td>
<td>$8x - 3x - 9ax$</td>
</tr>
<tr>
<td>$(9x + 10x + 3)$</td>
<td>$36x + a - 12ax$</td>
</tr>
<tr>
<td>$(9x - 10x + 3)$</td>
<td>$-24x - a - 25ax$</td>
</tr>
<tr>
<td>$(9x - 10x - 3)$</td>
<td>$-36x - a - 12ax$</td>
</tr>
<tr>
<td>$(4x + 3x + 1)$</td>
<td>$-12x + 2x - 10ax$</td>
</tr>
<tr>
<td>$(4x - 3x + 1)$</td>
<td>$-12a + 2x - 10ax$</td>
</tr>
<tr>
<td>$(4x - 3x - 1)$</td>
<td>$-12a - 2x - 10ax$</td>
</tr>
<tr>
<td>$(4x - 3x + 1)$</td>
<td>$-12a + 2x - 10ax$</td>
</tr>
</tbody>
</table>

This is the required middle term.

Thus, $8x^2 - 10x - 3 = (4x + 1)(2x - 3)$ or $(2x - 3)(4x + 1)$.

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The Difference of Two Squares

- If $A$ and $B$ are real numbers, variables, or algebraic expressions, then
- In words: The difference of the squares of two terms factors as the product of a sum and the difference of those terms.

Text Example

- Factor: $81x^2 - 49$

Solution:

$$81x^2 - 49 = (9x)^2 - 7^2 = (9x + 7)(9x - 7).$$

Factoring Perfect Square Trinomials

Let $A$ and $B$ be real numbers, variables, or algebraic expressions,
1. $A^2 + 2AB + B^2 = (A + B)^2$
2. $A^2 - 2AB + B^2 = (A - B)^2$
Text Example

- Factor: $x^2 + 6x + 9$.

Solution:

$$x^2 + 6x + 9 = x^2 + 2x \cdot 3 + 3^2 = (x + 3)^2$$

Text Example

- Factor: $25x^2 - 60x + 36$.

Solution:

$$25x^2 - 60x + 36 = (5x)^2 - 2 \cdot 5x \cdot 6 + 6^2 = (x + 3)^2$$

Factoring the Sum and Difference of 2 Cubes

<table>
<thead>
<tr>
<th>Type</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^3 + B^3$</td>
<td>$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$</td>
</tr>
<tr>
<td>$A^3 - B^3$</td>
<td>$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$</td>
</tr>
</tbody>
</table>
A Strategy for Factoring a Polynomial
1. If there is a common factor, factor out the GCF.
2. Determine the number of terms in the polynomial and try factoring as follows:
   a) If there are two terms, can the binomial be factored by one of the special forms including difference of two squares, sum of two cubes, or difference of two cubes?
   b) If there are three terms, is the trinomial a perfect square trinomial? If the trinomial is not a perfect square trinomial, try factoring by trial and error.
   c) If there are four or more terms, try factoring by grouping.
3. Check to see if any factors with more than one term in the factored polynomial can be factored further. If so, factor completely.

Factor:
\[x^3 – 5x^2 – 4x + 20\]

Solution
\[= (x^3 – 5x^2) + (-4x + 20)\] Group the terms with common factors.
\[= x^2(x – 5) – 4(x – 5)\] Factor from each group.
\[= (x – 5)(x^2 – 4)\] Factor out the common binomial factor, \((x – 5)\).
\[= (x – 5)(x + 2)(x – 2)\] Factor completely by factoring \(x^2 – 4\) as the difference of two squares.

Example

Factor: \(x^2 – 5x^2 – 4x + 20\)

Solution
\[x^2 – 5x^2 – 4x + 20\] Group the terms with common factors.
\[= (x^2 – 5x^2) + (-4x + 20)\] Factor from each group.
\[= x(x – 5) – 4(x – 5)\] Factor out the common binomial factor, \((x – 5)\).
\[= (x – 5)(x^2 – 4)\] Factor completely by factoring \(x^2 – 4\) as the difference of two squares.

Factoring Polynomials