Section 3-6

General Power Rule (Chain Rule)

➤ Chain Rule: Power Rule
➤ Combining Rules of Differentiation

In this section we develop a rule for differentiating powers of functions—a special case of the very important chain rule, which we will return to in Chapter 5. Also, for the first time, we will encounter some product forms that cannot be simplified by multiplication and must be differentiated by the power rule.

➤ Chain Rule: Power Rule

We have already made extensive use of the power rule,

\[
\frac{d}{dx} x^n = nx^{n-1}
\]  

(1)

Now we want to generalize this rule so that we can differentiate functions of the form \([u(x)]^n\), where \(u(x)\) is a differentiable function. Is rule (1) still valid if we replace \(x\) with a function \(u(x)\)?

Let \(u(x) = 2x^2\) and \(f(x) = [u(x)]^3 = 8x^6\). Which of the following is \(f'(x)\)?

- (A) \(3[u(x)]^2\)
- (B) \(3[u'(x)]^2\)
- (C) \(3[u(x)]^2u'(x)\)
The calculations in Explore–Discuss 1 show that we cannot generalize the power rule simply by replacing \( x \) with \( u(x) \) in equation (1).

How can we find a formula for the derivative of \( [u(x)]^n \), where \( u(x) \) is an arbitrary differentiable function? Let’s begin by considering the derivatives of \( [u(x)]^2 \) and \( [u(x)]^3 \) to see if a general pattern emerges. Since \( [u(x)]^2 = u(x)u(x) \), we use the product rule to write

\[
\frac{d}{dx} [u(x)]^2 = \frac{d}{dx} [u(x)u(x)]
\]

\[
= u(x)u'(x) + u(x)u'(x)
\]

\[
= 2u(x)u'(x)
\]

Since \( [u(x)]^3 = [u(x)]^2 u(x) \), we now use the product rule and the result in equation (2) to write

\[
\frac{d}{dx} [u(x)]^3 = \frac{d}{dx} ([u(x)]^2 u(x))
\]

\[
= [u(x)]^2 \frac{d}{dx} u(x) + u(x) \frac{d}{dx} [u(x)]^2
\]

\[
= [u(x)]^2 u'(x) + u(x) [2u(x)u'(x)]
\]

\[
= 3[u(x)]^2 u'(x)
\]

Continuing in this fashion, it can be shown that

\[
\frac{d}{dx} [u(x)]^n = n[u(x)]^{n-1} u'(x) \quad \text{n a positive integer} \tag{3}
\]

Using more advanced techniques, formula (3) can be established for all real numbers \( n \). Thus, we have the general power rule:

**Theorem 1** General Power Rule

If \( u(x) \) is a differentiable function, \( n \) is any real number, and

\[ y = f(x) = [u(x)]^n \]

then

\[ f'(x) = n[u(x)]^{n-1} u'(x) \]

This rule is often written more compactly as

\[ y' = nu^{n-1}u' \quad \text{or} \quad \frac{d}{dx} u^n = nu^{n-1} \frac{d}{dx} u \quad \text{where} \ u = u(x) \]

The general power rule is a special case of a very important and useful differentiation rule called the **chain rule**. In essence, the chain rule will enable us to differentiate a composition form \( f[g(x)] \) if we know how to differentiate \( f(x) \) and \( g(x) \). We defer a complete discussion of the chain rule until Chapter 5.

**Example 1** Using the General Power Rule

Find the indicated derivatives:

(A) \( f'(x) \) if \( f(x) = (3x + 1)^4 \)

(B) \( y' \) if \( y = (x^3 + 4)^7 \)

(C) \( \frac{d}{dt} (t^2 + t + 4)^3 \)

(D) \( \frac{dh}{dw} \) if \( h(w) = \sqrt{3 - w} \)
Solution

(A)  
\[ f(x) = (3x + 1)^4 \]
\[ f'(x) = 4(3x + 1)^3(3x + 1)' \]
\[ = 4(3x + 1)^3 3 \]
\[ = 12(3x + 1)^3 \]

(B)  
\[ y = (x^2 + 4)^7 \]
\[ y' = 7(x^2 + 4)^6(x^2 + 4)' \]
\[ = 7(x^2 + 4)^6 3x^2 \]
\[ = 21x^2(x^2 + 4)^6 \]

(C)  
\[ \frac{d}{dt} \frac{1}{(t^2 + t + 4)^3} \]
\[ = \frac{d}{dt} (t^2 + t + 4)^{-3} \]
\[ = -3(t^2 + t + 4)^{-4}(t^2 + t + 4)' \]
\[ = -3(t^2 + t + 4)^{-4}(2t + 1) \]
\[ = \frac{-3(2t + 1)}{(t^2 + t + 4)^4} \]

(D)  
\[ h(w) = \sqrt{3 - w} = (3 - w)^{1/2} \]
\[ h'(w) = \frac{1}{2} (3 - w)^{-1/2}(3 - w)' \]
\[ = \frac{1}{2} (3 - w)^{-1/2}(-1) \]
\[ = \frac{-1}{2(3 - w)^{1/2}} \]


Matched Problem 1

Find the indicated derivatives:

(A) \[ h'(x) \quad \text{if} \quad h(x) = (5x + 2)^3 \]
(B) \[ y' \quad \text{if} \quad y = (x^4 - 5)^5 \]
(C) \[ \frac{d}{dt} \frac{1}{(t^2 + 4)^2} \]
(D) \[ \frac{dg}{dw} \quad \text{if} \quad g(w) = \sqrt{4 - w} \]

Notice that we used two steps to differentiate each function in Example 1. First, we applied the general power rule; then we found \( \frac{du}{dx} \). As you gain experience with the general power rule, you may want to combine these two steps. If you do this, be certain to multiply by \( \frac{du}{dx} \). For example,

\[ \frac{d}{dx} (x^5 + 1)^4 = 4(x^5 + 1)^3x^4 \quad \text{Correct} \]
\[ \frac{d}{dx} (x^5 + 1)^4 \neq 4(x^5 + 1)^3 \quad \text{du/dx = 5x^4 is missing} \]
If we let $u(x) = x$, then $du/dx = 1$, and the general power rule reduces to the (ordinary) power rule discussed in Section 3-4. Compare the following:

$$\frac{d}{dx} x^n = nx^{n-1} \quad \text{Yes — power rule}$$

$$\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx} \quad \text{Yes — general power rule}$$

$$\frac{d}{dx} u^n \neq nu^{n-1} \quad \text{Unless } u(x) = x + k \text{ so that } du/dx = 1$$

### Combining Rules of Differentiation

The following examples illustrate the use of the general power rule in combination with other rules of differentiation.

#### Example 2

**Tangent Lines** Find the equation of the line tangent to the graph of $f$ at $x = 2$ for $f(x) = x^2\sqrt{2x + 12}$.

**Solution**

$$f(x) = x^2\sqrt{2x + 12}$$

$$= x^2(2x + 12)^{1/2} \quad \text{Apply the product rule.}$$

$$f'(x) = x^2 \frac{d}{dx} (2x + 12)^{1/2} + (2x + 12)^{1/2} \frac{d}{dx} x^2$$

$$= x^2 \left[ \frac{1}{2} (2x + 12)^{-1/2} \right] (2) + (2x + 12)^{1/2} (2x)$$

$$= \frac{x^2}{\sqrt{2x + 12}} + 2x\sqrt{2x + 12}$$

$$f'(2) = \frac{4}{\sqrt{16}} + 4\sqrt{16} = 1 + 16 = 17$$

$$f(2) = 4\sqrt{16} = 16$$

$$(x_1, y_1) = (2, f(2)) = (2, 16) \quad \text{Point}$$

$$m = f'(2) = 17 \quad \text{Slope}$$

$$y - 16 = 17(x - 2) \quad y - y_1 = m(x - x_1)$$

$$y = 17x - 18 \quad \text{Tangent line}$$

**Matched Problem 2**

Find $f'(x)$ and the equation of the line tangent to the graph of $f$ at $x = 3$ for $f(x) = x\sqrt{15 - 2x}$.

#### Example 3

**Tangent Lines** Find the value(s) of $x$ where the tangent line is horizontal for

$$f(x) = \frac{x^3}{(2 - 3x)^5}$$
Solution

Use the quotient rule:

\[
\frac{d}{dx}(2 - 3x)^5 = \frac{(2 - 3x)^5 \cdot \frac{d}{dx}x^3 - x^3 \cdot \frac{d}{dx}(2 - 3x)^5}{(2 - 3x)^{10}}
\]

\[
= \frac{(2 - 3x)^5 \cdot 3x^2 - x^3 \cdot 5(2 - 3x)^4(-3)}{(2 - 3x)^{10}}
\]

\[
= \frac{(2 - 3x)^5 \cdot 3x^2[(2 - 3x) + 5x]}{(2 - 3x)^{10}}
\]

\[
= \frac{3x^2(2 + 2x)}{(2 - 3x)^6} = \frac{6x^2(x + 1)}{(2 - 3x)^6}
\]

Since a fraction is 0 when the numerator is 0 and the denominator is not, we see that \( f'(x) = 0 \) at \( x = -1 \) and \( x = 0 \). Thus, the graph of \( f \) will have horizontal tangent lines at \( x = -1 \) and \( x = 0 \).

Matched Problem 3

Find the value(s) of \( x \) where the tangent line is horizontal for

\[
f(x) = \frac{x^3}{(3x - 2)^2}
\]
As Example 4 illustrates, any quotient can be converted to a product and differentiated by the product rule. However, if the derivative must be simplified, it is usually easier to use the quotient rule. (Compare the algebraic simplifications in Example 4 with those in Example 3.) There is one special case where using negative exponents is the preferred method—a fraction whose numerator is a constant.

**Example 5**

**Alternate Methods of Differentiation** Find \( f'(x) \) two ways for

\[
f(x) = \frac{4}{(x^2 + 9)^3}
\]

**Solution**

**Method 1.** Use the quotient rule:

\[
f'(x) = \frac{\frac{d}{dx}(x^2 + 9)^3 \cdot 4 - 4 \cdot \frac{d}{dx} (x^2 + 9)^3}{(x^2 + 9)^6}
\]

\[
= \frac{(x^2 + 9)^3(0) - 4[3(x^2 + 9)^2(2x)]}{(x^2 + 9)^6}
\]

\[
= \frac{-24x(x^2 + 9)^2}{(x^2 + 9)^6} = \frac{-24x}{(x^2 + 9)^4}
\]

**Method 2.** Rewrite as a product, and use the general power rule:

\[
f(x) = \frac{4}{(x^2 + 9)^3} = 4(x^2 + 9)^{-3}
\]

\[
f'(x) = 4(-3)(x^2 + 9)^{-4}(2x)
\]

\[
= \frac{-24x}{(x^2 + 9)^4}
\]

Which method do you prefer?

**Matched Problem 5**

Find \( f'(x) \) two ways for \( f(x) = \frac{5}{(x^3 + 1)^2} \).

1. (A) 15(x^2 + 2)^2
   (B) 20x^3(x^4 - 5)^4
   (C) -4t/(t^2 + 4)^3
   (D) -1/(2\sqrt{4 - \pi})
2. \( f'(x) = \sqrt{15} - 2x - \frac{x}{\sqrt{15} - 2x} \); \( y = 2x + 3 \)
3. \( x = 0, x = 2 \)
4. \(-6x^2(3x - 2)^{-3} + 3x^2(3x - 2)^{-2} = \frac{3x^2(x - 2)}{(3x - 2)^3} \)
5. \(-30x^2 \)

\( (x^3 + 1)^3 \)