SECTION 3.1 • Systems of Linear Equations in Two Variables

Objectives

1. Determine whether an ordered pair is a solution of a system of linear equations.
2. Solve systems of linear equations by graphing.
3. Solve systems of linear equations by substitution.
4. Solve systems of linear equations by addition.
5. Select the most efficient method for solving a system of linear equations.
6. Identify systems that do not have exactly one ordered-pair solution.

Women now attain higher levels of education than ever before and a majority of women choose careers in the labor force rather than homemaking. One of the most important developments in the work force has been the dramatic increase in the number of women, at approximately 4% per year. By contrast, the percentage of men is decreasing by 1% per year. These changes are illustrated in Figure 3.1. If current trends continue, the graphs will intersect. The intersection point reveals when the percentage of women and men in the labor force will be equal.

Projections frequently involve graphs that intersect. In this section, you will learn three methods for finding intersection points. With these methods, you can use mathematical models to determine when variable quantities, such as the percentage of women and men in the work force, will be equal.

Systems of Linear Equations and Their Solutions

We have seen that all equations in the form \( Ax + By = C \) are straight lines when graphed. Two such equations are called a system of linear equations. A solution of a system of linear equations is an ordered pair that satisfies all equations in the system. For example, \((3, 4)\) satisfies the system

\[
\begin{align*}
x + y &= 7 \quad (3 + 4 \text{ is, indeed, 7}) \\
x - y &= -1 \quad (3 - 4 \text{ is, indeed, } -1)
\end{align*}
\]

Thus, \((3, 4)\) satisfies both equations and is a solution of the system. The solution can be described by saying that \(x = 3\) and \(y = 4\). The solution can also be described using set notation. The solution set of the system is \(\{(3, 4)\}\) — that is, the set consisting of the ordered pair \((3, 4)\).

A system of linear equations can have exactly one solution, no solution, or infinitely many solutions. We begin with systems with exactly one solution.

**Example 1**

**Determining Whether an Ordered Pair Is a Solution of a System**

Determine whether \((-5, -6)\) is a solution of the system

\[
\begin{align*}
2x - y &= -4 \\
3x - 5y &= 15
\end{align*}
\]
SOLVING SYSTEMS OF TWO LINEAR EQUATIONS IN TWO VARIABLES, x AND y, BY GRAPHING

1. Graph the first equation.
2. Graph the second equation on the same set of axes.
3. If the lines representing the two graphs intersect at a point, determine the coordinates of this point of intersection. The ordered pair is the solution to the system.
4. Check the solution in both equations.

EXAMPLE 2 Solving a Linear System by Graphing

Solve by graphing:

\[ y = -x - 1 \]
\[ 4x - 3y = 24. \]

**SOLUTIION**

**Step 1.** Graph the first equation. We use the y-intercept and slope to graph \( y = -x - 1 \).

The slope is \(-1\). The y-intercept is \(-1\).

The graph of the linear function is shown as a blue line in Figure 3.2.

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Solve systems of linear equations by substitution.

**USING TECHNOLOGY**
A graphing utility can be used to solve the system in Example 2. Solve each equation for y, graph the equations, and use the intersection feature. The utility displays the solution as (3, -4).

Step 2. **Graph the second equation on the same axes.** We use intercepts to graph 4x - 3y = 24.

\[
\begin{align*}
\text{x-intercept:} & \quad \text{Set } y = 0.0 \quad \quad \text{y-intercept:} \quad \text{Set } x = 0.0 \\
4x - 3 \cdot 0 &= 24 & 4 \cdot 0 - 3y &= 24 \\
4x &= 24 & -3y &= 24 \\
x &= 6 & y &= -8 \\
\end{align*}
\]

The x-intercept is 6, so the line passes through (6, 0). The y-intercept is -8, so the line passes through (0, -8). The graph of 4x - 3y = 24 is shown as a red line in Figure 3.2.

Step 3. **Determine the coordinates of the intersection point.** This ordered pair is the system’s solution.

Step 4. **Check the solution in both equations.**

Check (3, -4) in

\[
\begin{align*}
y &= -x - 1: \quad & 4x - 3y &= 24: \\
-4 &= -3 - 1 & 4(3) - 3(-4) &= 24 \\
-4 &= -4, \quad \text{true} & 12 + 12 &= 24 \\
& & 24 &= 24, \quad \text{true} \\
\end{align*}
\]

Because both equations are satisfied, (3, -4) is the solution and \((3, -4)\) is the solution set.

**CHECK POINT 2** Solve by graphing:

\[
\begin{align*}
y &= -2x + 6 \\
2x - y &= -2.
\end{align*}
\]

Eliminating a Variable Using the Substitution Method Finding the solution to a linear system by graphing equations may not be easy to do. For example, a solution of \((-\frac{2}{3}, \frac{15}{8})\) would be difficult to “see” as an intersection point on a graph.

Let’s consider a method that does not depend on finding a system’s solution visually: the substitution method. This method involves converting the system to one equation in one variable by an appropriate substitution.

**EXAMPLE 3** **Solving a System by Substitution**

Solve by the substitution method:

\[
y = -2x + 4 \\
7x - 2y = 3.
\]

**SOLUTION**

Step 1. **Solve either of the equations for one variable in terms of the other.** This step has already been done for us. The first equation, \(y = -2x + 4\), has \(y\) solved in terms of \(x\).

Step 2. **Substitute the expression from step 1 into the other equation.** We substitute the expression \(-2x + 4\) for \(y\) in the other equation:

\[
y = -2x + 4 \\
7x - 2y = 3. \quad \text{Substitute } -2x + 4 \text{ for } y.
\]

This gives us an equation in one variable, namely

\[
7x - 2(-2x + 4) = 3.
\]

The variable \(y\) has been eliminated.
Chapter 3 • Systems of Linear Equations

Step 3. Solve the resulting equation containing one variable.

\[ 7x - 2(-2x + 4) = 3 \]
This is the equation containing one variable.
\[ 7x + 4x - 8 = 3 \]
Apply the distributive property.
\[ 11x - 8 = 3 \]
Combine like terms.
\[ 11x = 11 \]
Add 8 to both sides.
\[ x = 1 \]
Divide both sides by 11.

Step 4. Back-substitute the obtained value into one of the original equations. We now know that the \( x \)-coordinate of the solution is 1. To find the \( y \)-coordinate, we back-substitute the \( x \)-value into either original equation. We will use

\[ y = -2x + 4. \]

Substitute 1 for \( x \).

\[ y = -2 \cdot 1 + 4 = -2 + 4 = 2 \]

With \( x = 1 \) and \( y = 2 \), the proposed solution is \((1, 2)\).

Step 5. Check the proposed solution in both of the system’s given equations. Replace \( x \) with 1 and \( y \) with 2.

\[ y = -2x + 4 \quad 7x - 2y = 3 \]
\[ 2 \overset{?}{=} -2 \cdot 1 + 4 \quad 7(1) - 2(2) \overset{?}{=} 3 \]
\[ 2 \overset{?}{=} -2 + 4 \quad 7 - 4 \overset{?}{=} 3 \]
\[ 2 = 2, \text{ true} \quad 3 = 3, \text{ true} \]

The pair \((1, 2)\) satisfies both equations. The solution is \((1, 2)\) and the system’s solution set is \(\{(1, 2)\}\).

CHECK POINT 3 Solve by the substitution method:

\[ y = 3x - 7 \]
\[ 5x - 2y = 8. \]

Before considering additional examples, let’s summarize the steps used in the substitution method.

STUDY TIP

In step 1, you can choose which variable to isolate in which equation. If possible, solve for a variable whose coefficient is 1 or \(-1\) to avoid working with fractions.

SOLVING LINEAR SYSTEMS BY SUBSTITUTION

1. Solve either of the equations for one variable in terms of the other. (If one of the equations is already in this form, you can skip this step.)
2. Substitute the expression found in step 1 into the other equation. This will result in an equation in one variable.
3. Solve the equation containing one variable.
4. Back-substitute the value found in step 3 into one of the original equations. Simplify and find the value of the remaining variable.
5. Check the proposed solution in both of the system’s given equations.
The equation from step 1, in which one variable is expressed in terms of the other, is equivalent to one of the original equations. It is often easiest to back-substitute an obtained value into this equation to find the value of the other variable. After obtaining both values, get into the habit of checking the ordered-pair solution in both equations of the system.
Eliminating a Variable Using the Addition Method  The substitution method is most useful if one of the given equations has an isolated variable. A third method for solving a linear system is the addition method. Like the substitution method, the addition method involves eliminating a variable and ultimately solving an equation containing only one variable. However, this time we eliminate a variable by adding the equations.

For example, consider the following equations:

\[3x - 4y = 11\]
\[-3x + 2y = -7.\]

When we add these two equations, the \(x\)-terms are eliminated. This occurs because the coefficients of the \(x\)-terms, 3 and \(-3\), are opposites (additive inverses) of each other:

\[\begin{align*}
3x - 4y &= 11 \\
-3x + 2y &= -7
\end{align*}\]

**Add:**
\[0x - 2y = 4\]
\[-2y = 4\]
\[y = -2.\]

Now we can back-substitute \(-2\) for \(y\) into one of the original equations to find \(x\). It does not matter which equation you use; you will obtain the same value for \(x\) in either case. If we use either equation, we can show that \(x = 1\) and the solution \((1, -2)\) satisfies both equations in the system.

When we use the addition method, we want to obtain two equations whose sum is an equation containing only one variable. The key step is to obtain, for one of the variables, **coefficients that differ only in sign**. To do this, we may need to multiply one or both equations by some nonzero number so that the coefficients of one of the variables, \(x\) or \(y\), become opposites. Then when the two equations are added, this variable is eliminated.

**Example 5** Solve a System by the Addition Method

Solve by the addition method:

\[3x + 4y = -10\]
\[5x - 2y = 18.\]

**Solution** We must rewrite one or both equations in equivalent forms so that the coefficients of the same variable (either \(x\) or \(y\)) are opposites of each other. Consider the terms in \(y\) in each equation, that is, \(4y\) and \(-2y\). To eliminate \(y\), we can multiply each term of the second equation by 2 and then add equations:

\[\begin{align*}
3x + 4y &= -10 \\
5x - 2y &= 18
\end{align*}\]

**Add:**
\[10x + 0y = 36\]
\[13x = 36\]
\[x = 2\]

Divide both sides by 13 and solve for \(x\).
Thus, \( x = 2 \). We back-substitute this value into either one of the given equations. We'll use the first one.

\[
\begin{align*}
3x + 4y &= -10 & \text{This is the first equation in the given system.} \\
3(2) + 4y &= -10 & \text{Substitute 2 for } x. \\
6 + 4y &= -10 & \text{Multiply.} \\
4y &= -16 & \text{Subtract 6 from both sides.} \\
y &= -4 & \text{Divide both sides by 4.}
\end{align*}
\]

We see that \( x = 2 \) and \( y = -4 \). The ordered pair \((2, -4)\) can be shown to satisfy both equations in the system. Consequently, the solution is \((2, -4)\) and the solution set is \( \{ (2, -4) \} \).

**SOLVING LINEAR SYSTEMS BY ADDITION**

1. If necessary, rewrite both equations in the form \( Ax + By = C \).
2. If necessary, multiply either equation or both equations by appropriate nonzero numbers so that the sum of the \( x \)-coefficients or the sum of the \( y \)-coefficients is 0.
3. Add the equations in step 2. The sum is an equation in one variable.
4. Solve the equation in one variable.
5. Back-substitute the value obtained in step 4 into either of the given equations and solve for the other variable.
6. Check the solution in both of the original equations.

**CHECK POINT 5** Solve by the addition method:

\[
\begin{align*}
4x - 7y &= -16 \\
2x + 5y &= 9,
\end{align*}
\]

**EXAMPLE 6** Solving a System by the Addition Method

Solve by the addition method:

\[
\begin{align*}
7x &= 5 - 2y \\
3y &= 16 - 2x.
\end{align*}
\]

**SOLUTION**

Step 1. **Rewrite both equations in the form** \( Ax + By = C \). We first arrange the system so that variable terms appear on the left and constants appear on the right. We obtain

\[
\begin{align*}
7x + 2y &= 5 & \text{Add } 2y \text{ to both sides of the first equation.} \\
2x + 3y &= 16 & \text{Add } 2x \text{ to both sides of the second equation.}
\end{align*}
\]
Step 2. If necessary, multiply either equation or both equations by appropriate numbers so that the sum of the x-coefficients or the sum of the y-coefficients is 0. We can eliminate x or y. Let’s eliminate y by multiplying the first equation by 3 and the second equation by –2.

\[
\begin{align*}
7x + 2y &= 5 \\
2x + 3y &= 16
\end{align*}
\]

Multiply by 3.

\[
\begin{align*}
21x + 6y &= 15 \\
-4x - 6y &= -32
\end{align*}
\]

Step 3. Add the equations.

\[
\begin{align*}
17x + 0y &= -17 \\
17x &= -17
\end{align*}
\]

Step 4. Solve the equation in one variable. We solve \(17x = -17\) by dividing both sides by 17.

\[
\frac{17x}{17} = \frac{-17}{17}
\]

Divide both sides by 17.

\[
x = -1
\]

Simplify.

Step 5. Back-substitute and find the value for the other variable. We can back-substitute –1 for x into either one of the given equations. We’ll use the second one.

\[
3y = 16 - 2x
\]

This is the second equation in the given system.

\[
3y = 16 - 2(-1)
\]

Substitute –1 for x.

\[
3y = 16 + 2
\]

Multiply.

\[
3y = 18
\]

Add.

\[
y = 6
\]

Divide both sides by 3.

With \(x = -1\) and \(y = 6\), the proposed solution is \((-1, 6)\).

Step 6. Check. Take a moment to show that \((-1, 6)\) satisfies both given equations. The solution is \((-1, 6)\) and the solution set is \{\((-1, 6)\}\).

\[
\begin{align*}
3x &= 2 - 4y \\
5y &= -1 - 2x
\end{align*}
\]

Some linear systems have solutions that are not integers. If the value of one variable turns out to be a “messy” fraction, back-substitution might lead to cumbersome arithmetic. If this happens, you can return to the original system and use addition to find the value of the other variable.

**Example 7** Solving a System by the Addition Method

Solve by the addition method:

\[
\begin{align*}
\frac{x}{2} - 5y &= 32 \\
\frac{3x}{2} - 7y &= 45
\end{align*}
\]

**Solution**

Step 1. Rewrite both equations in the form \(Ax + By = C\). Although each equation is already in this form, the coefficients of \(x\) are not integers. There is less
chance for error if the coefficients for \( x \) and \( y \) in \( Ax + By = C \) are integers. Consequently, we begin by clearing fractions. Multiply both sides of each equation by 2.

\[
\begin{align*}
\frac{x}{2} - 5y &= 32 \quad \text{Multiply by 2,} \\
3x - 7y &= 45 \quad \text{Multiply by 2,}
\end{align*}
\]

\[
\begin{align*}
x - 10y &= 64 \quad \text{Multiply by 3,} \\
3x - 14y &= 90 \quad \text{No change}
\end{align*}
\]

**Step 2.** If necessary, multiply either equation or both equations by appropriate numbers so that the sum of the \( x \)-coefficients or the sum of the \( y \)-coefficients is 0. We will eliminate \( x \). Multiply the first equation with integral coefficients by \(-3\) and leave the second equation unchanged.

\[
\begin{align*}
x - 10y &= 64 \quad \text{Multiply by } -3, \\
3x - 14y &= 90 \quad \text{No change}
\end{align*}
\]

A dd the equations.

\[
\begin{align*}
-3x + 30y &= -192 \\
3x - 14y &= 90
\end{align*}
\]

**Step 3.** \( A dd \) the equations. Add:

\[
0x + 16y = -102
\]

\[
16y = -102
\]

**Step 4.** Solve the equation in one variable. We solve 16\( y \) = -102 by dividing both sides by 16.

\[
\frac{16y}{16} = \frac{-102}{16} \quad \text{Divide both sides by 16.}
\]

\[
y = \frac{-102}{16} = -\frac{51}{8} \quad \text{Simplify.}
\]

**Step 5.** Back-substitute and find the value of the other variable. Back-substitution of \(-\frac{51}{8}\) for \( y \) into either of the given equations results in cumbersome arithmetic. Instead, let’s use the addition method on the system with integral coefficients to find the value of \( x \). Thus, we eliminate \( y \) by multiplying the first equation by \(-7\) and the second equation by 5.

\[
\begin{align*}
x - 10y &= 64 \quad \text{Multiply by } -7, \\
3x - 14y &= 90 \quad \text{Multiply by 5.}
\end{align*}
\]

A dd:

\[
8x = \frac{450}{2} \quad x = \frac{2}{8} = \frac{1}{4}
\]

With \( x = \frac{1}{4} \) and \( y = -\frac{51}{8} \), the proposed solution is \( \left( \frac{1}{4}, -\frac{51}{8} \right) \).

**Step 6.** Check. For this system, a calculator is helpful in showing that \( \left( \frac{1}{4}, -\frac{51}{8} \right) \) satisfies both of the original equations of the system. The solution is \( \left( \frac{1}{4}, -\frac{51}{8} \right) \) and the solution set is \( \left\{ \left( \frac{1}{4}, -\frac{51}{8} \right) \right\} \).

**CHECK POINT 7** Solve by the addition method:

\[
\begin{align*}
\frac{3x}{2} - 2y &= \frac{5}{2} \\
x - \frac{5y}{2} &= -\frac{3}{2}
\end{align*}
\]
Select the most efficient method for solving a system of linear equations.

Comparing the Three Solution Methods

The following chart compares the graphing, substitution, and addition methods for solving systems of linear equations in two variables. With increased practice, you will find it easier to select the best method for solving a particular linear system.

Comparing Solution Methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphing</td>
<td>You can see the solutions.</td>
<td>If the solutions do not involve integers or are too large to be seen on the graph, it’s impossible to tell exactly what the solutions are.</td>
</tr>
<tr>
<td>Substitution</td>
<td>Gives exact solutions. Easy to use if a variable is on one side by itself.</td>
<td>Solutions cannot be seen. Introduces extensive work with fractions when no variable has a coefficient of 1 or (-1).</td>
</tr>
<tr>
<td>Addition</td>
<td>Gives exact solutions. Easy to use if no variable has a coefficient of 1 or (-1).</td>
<td>Solutions cannot be seen.</td>
</tr>
</tbody>
</table>

Linear Systems Having No Solution or Infinitely Many Solutions

We have seen that a system of linear equations in two variables represents a pair of lines. The lines either intersect at one point, are parallel, or are identical. Thus, there are three possibilities for the number of solutions to a system of two linear equations.

The Number of Solutions to a System of Two Linear Equations

The number of solutions to a system of two linear equations in two variables is given by one of the following. (See Figure 3.3.)

<table>
<thead>
<tr>
<th>Number of Solutions</th>
<th>What This Means Graphically</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exactly one ordered-pair solution</td>
<td>The two lines intersect at one point.</td>
</tr>
<tr>
<td>No solution</td>
<td>The two lines are parallel.</td>
</tr>
<tr>
<td>Infinitely many solutions</td>
<td>The two lines are identical.</td>
</tr>
</tbody>
</table>

FIGURE 3.3 Possible graphs for a system of two linear equations in two variables
A linear system with no solution is called an **inconsistent system**. If you attempt to solve such a system by substitution or addition, you will eliminate both variables. A false statement, such as \(0 = 6\), will be the result.

### Example 8  A System with No Solution

Solve the system:
\[
\begin{align*}
3x - 2y &= 6 \\
6x - 4y &= 18.
\end{align*}
\]

**Solution** Because no variable is isolated, we will use the addition method. To obtain coefficients of \(x\) that differ only in sign, we multiply the first equation by \(-2\).

\[
\begin{align*}
3x - 2y &= 6 \quad \text{Multiply by} -2. \\
6x - 4y &= 18.
\end{align*}
\]

\[
\begin{align*}
-6x + 4y &= -12 \quad \text{No change} \\
6x - 4y &= 18
\end{align*}
\]

\[
\begin{align*}
0 &= 6 \\
\text{Add:}
\end{align*}
\]

The false statement \(0 = 6\) indicates that the system is inconsistent and has no solution. The solution set is the empty set, \(\emptyset\).

The lines corresponding to the two equations in Example 8 are shown in Figure 3.4. The lines are parallel and have no point of intersection.

### Discover for Yourself

Show that the graphs of \(3x - 2y = 6\) and \(6x - 4y = 18\) must be parallel lines by solving each equation for \(y\). What is the slope and \(y\)-intercept for each line? What does this mean? If a linear system is inconsistent, what must be true about the slopes and \(y\)-intercepts for the system’s graphs?

### Check Point 8

Solve the system:
\[
\begin{align*}
5x - 2y &= 4 \\
-10x + 4y &= 7
\end{align*}
\]

A linear system that has at least one solution is called a **consistent system**. Lines that intersect and lines that coincide both represent consistent systems. If the lines coincide, then the consistent system has infinitely many solutions, represented by every point on the coinciding lines.

The equations in a linear system with infinitely many solutions are called **dependent**. If you attempt to solve such a system by substitution or addition, you will eliminate both variables. However, a true statement, such as \(10 = 10\), will be the result.

### Example 9  A System with Infinitely Many Solutions

Solve the system:
\[
\begin{align*}
\frac{y}{x} &= 3x - 2 \\
15x - 5y &= 10.
\end{align*}
\]
SOLUTION  Because the variable $y$ is isolated in $y = 3x - 2$, the first equation, we can use the substitution method. We substitute the expression for $y$ into the second equation.

$$y = \frac{3x - 2}{15x - 5y = 10}$$

Substitute $3x - 2$ for $y$.

$$15x - 5(3x - 2) = 10$$
The substitution results in an equation in one variable.

$$15x - 15x + 10 = 10$$
Apply the distributive property.

$$10 = 10$$
Simplify.

In our final step, both variables have been eliminated and the resulting statement, $10 = 10$, is true. This true statement indicates that the system has infinitely many solutions. The solution set consists of all points $(x, y)$ lying on either of the coinciding lines, $y = 3x - 2$ or $15x - 5y = 10$, as shown in Figure 3.5.

We express the solution set for the system in one of two equivalent ways:

$$(x, y) \ | \ y = 3x - 2 \quad \text{or} \quad (x, y) \ | \ 15x - 5y = 10.$$