Objectives

1. Solve problems using systems of equations.
2. Use functions to model revenue, cost, and profit.
3. Perform a break-even analysis.

Driving through your neighborhood, you see kids selling lemonade. Would it surprise you to know that this activity can be analyzed using functions and systems of equations? By doing so, you will view profit and loss in the business world in a new way. In this section, we use systems of equations to solve problems and model business ventures.

A Strategy for Solving Word Problems Using Systems of Equations. When we solved problems in Chapter 1, we let \( x \) represent a quantity that was unknown. Problems in this section involve two unknown quantities. We will let \( x \) and \( y \) represent these quantities. We then translate from the verbal conditions of the problem into a system of linear equations.

Copyright © 2006 Pearson Education, Inc., publishing as Pearson Prentice Hall
EXAMPLE 1  Solving a Problem Involving Energy Efficiency of Building Materials

A heat-loss survey by an electric company indicated that a wall of a house containing 40 square feet of glass and 60 square feet of plaster lost 1920 Btu (British thermal units) of heat. A second wall containing 10 square feet of glass and 100 square feet of plaster lost 1160 Btu of heat. Determine the heat lost per square foot for the glass and for the plaster.

SOLUTION

Step 1. Use variables to represent unknown quantities.
Let $x$ = the heat lost per square foot for the glass.
Let $y$ = the heat lost per square foot for the plaster.

Step 2. Write a system of equations describing the problem's conditions. The heat loss for each wall is the heat lost by the glass plus the heat lost by the plaster.
One wall containing 40 square feet of glass and 60 square feet of plaster lost 1920 Btu of heat.

$$40 \cdot x + 60 \cdot y = 1920$$

A second wall containing 10 square feet of glass and 100 square feet of plaster lost 1160 Btu of heat.

$$10 \cdot x + 100 \cdot y = 1160$$

Step 3. Solve the system and answer the problem's question. The system

$$40x + 60y = 1920$$
$$10x + 100y = 1160$$

can be solved by addition. We'll multiply the second equation by $-4$ and then add equations to eliminate $x$.

$$40x + 60y = 1920$$
$$10x + 100y = 1160$$

Multiply by $-4$.

$$-40x - 400y = -4640$$
$$340y = -2720$$

Add:

$$y = -2720$$
$$340$$

$$y = 8$$

Now we can find the value of $x$ by back-substituting 8 for $y$ in either of the system's equations.

$$10x + 100y = 1160$$

We'll use the second equation.

$$10x + 100(8) = 1160$$

Back-substitute 8 for $y$.

$$10x = 360$$

Multiply.

$$x = 36$$

Divide both sides by 10.
We see that \( x = 36 \) and \( y = 8 \). Because \( x \) represents heat lost per square foot for the glass and \( y \) for the plaster, the glass lost 36 Btu of heat per square foot and the plaster lost 8 Btu per square foot.

**Step 4. Check the proposed solution in the original wording of the problem.** The problem states that the wall with 40 square feet of glass and 60 square feet of plaster lost 1920 Btu.

\[
40(36) + 60(8) = 1440 + 480 = 1920 \text{ Btu of heat}
\]

Our proposed solution checks with the first statement. The problem also states that the wall with 10 square feet of glass and 100 square feet of plaster lost 1160 Btu.

\[
10(36) + 100(8) = 360 + 800 = 1160 \text{ Btu of heat}
\]

Our proposed solution also checks with the second statement.

**CHECK POINT 1** Two Snickers bars and one Think Thin bar contain 70 grams of carbohydrates. One Snickers bar and two Think Thin bars contain 35 grams of carbohydrates. Find the carbohydrate content in each item.

Next, we will solve problems involving investments, mixtures, and motion with systems of equations. We will continue using our four-step problem-solving strategy. We will also use tables to help organize the information in the problems.

**Dual Investments with Simple Interest** Simple interest involves interest calculated only on the amount of money that we invest, called the **principal**. The formula \( I = Pr \) is used to find the simple interest, \( I \), earned for one year when the principal, \( P \), is invested at an annual interest rate, \( r \). Dual investment problems involve different amounts of money in two or more investments, each paying a different rate.

**EXAMPLE 2 Solving a Dual Investment Problem**

Your grandmother needs your help. She has $50,000 to invest. Part of this money is to be invested in noninsured bonds paying 15% annual interest. The rest of this money is to be invested in a government-insured certificate of deposit paying 7% annual interest. She told you that she requires $6000 per year in extra income from both of these investments. How much money should be placed in each investment?

**SOLUTION**

**Step 1. Use variables to represent unknown quantities.**
Let \( x \) = the amount invested in the 15% noninsured bonds.
Let \( y \) = the amount invested in the 7% certificate of deposit.

**Step 2. Write a system of equations describing the problem’s conditions.** Because Grandma has $50,000 to invest,

\[
x + y = 50,000
\]
Furthermore, Grandma requires $6000 in total interest. We can use a table to organize the information in the problem and obtain a second equation.

<table>
<thead>
<tr>
<th>Principal (amount invested)</th>
<th>Interest rate</th>
<th>Interest earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>15% Investment</td>
<td>x</td>
<td>0.15</td>
</tr>
<tr>
<td>7% Investment</td>
<td>y</td>
<td>0.07</td>
</tr>
</tbody>
</table>

The interest for the two investments combined must be $6000.

\[ 0.15x + 0.07y = 6000 \]

**Step 3. Solve the system and answer the problem’s question.** The system

\[ x + y = 50,000 \]
\[ 0.15x + 0.07y = 6000 \]

can be solved by substitution or addition. Substitution works well because both variables in the first equation have coefficients of 1. Addition also works well; if we multiply the first equation by \(-0.15\) or \(-0.07\), adding equations will eliminate a variable. We will use addition.

\[ x + y = 50,000 \rightarrow Multiply by \(-0.07\). \]
\[ 0.15x + 0.07y = 6000 \rightarrow \text{No change} \]
\[ 0.08x = 2500 \]
\[ x = 31,250 \]

Because \(x\) represents the amount that should be invested at 15%, Grandma should place $31,250 in 15% noninsured bonds. Now we can find \(y\), the amount that she should place in the 7% certificate of deposit. We do so by back-substituting 31,250 for \(x\) in either of the system’s equations.

\[ x + y = 50,000 \quad \text{We’ll use the first equation.} \]
\[ 31,250 + y = 50,000 \quad \text{Back-substitute 31,250 for x.} \]
\[ y = 18,750 \quad \text{Subtract 31,250 from both sides.} \]

Because \(x = 31,250\) and \(y = 18,750\), Grandma should invest $31,250 at 15% and $18,750 at 7%.

**Step 4. Check the proposed answers in the original wording of the problem.** Has Grandma invested $50,000?

Yes, all her money was placed in the dual investments. Can she count on $6000 interest? The interest earned on $31,250 at 15% is \((31,250)(0.15)\), or $4687.50. The interest earned on $18,750 at 7% is \((18,750)(0.07)\), or $1312.50. The total interest is $4687.50 + $1312.50, or $6000, exactly as it should be. You’ve made your grandmother happy. (Now if you would just visit her more often . . . )
CHECK POINT 2  You inherited $5000 with the stipulation that for the first year the money had to be invested in two funds paying 9% and 11% annual interest. How much did you invest at each rate if the total interest earned for the year was $487?

Problems Involving Mixtures  Chemists and pharmacists often have to change the concentration of solutions and other mixtures. In these situations, the amount of a particular ingredient in the solution or mixture is expressed as a percentage of the total.

EXAMPLE 3  Solving a Mixture Problem

A chemist working on a flu vaccine needs to mix a 10% sodium-iodine solution with a 60% sodium-iodine solution to obtain 50 milliliters of a 30% sodium-iodine solution. How many milliliters of the 10% solution and of the 60% solution should be mixed?

SOLUTION

Step 1. Use variables to represent unknown quantities.
Let \( x \) = the number of milliliters of the 10% solution to be used in the mixture.
Let \( y \) = the number of milliliters of the 60% solution to be used in the mixture.

Step 2. Write a system of equations describing the problem's conditions. The situation is illustrated in Figure 3.6.

The chemist needs 50 milliliters of a 30% sodium-iodine solution. We form a table that shows the amount of sodium-iodine in each of the three solutions.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Number of milliliters</th>
<th>Percent of Sodium-Iodine</th>
<th>A mount of Sodium-Iodine</th>
</tr>
</thead>
<tbody>
<tr>
<td>10% Solution</td>
<td>( x )</td>
<td>10% = 0.1</td>
<td>0.1( x )</td>
</tr>
<tr>
<td>60% Solution</td>
<td>( y )</td>
<td>60% = 0.6</td>
<td>0.6( y )</td>
</tr>
<tr>
<td>30% Mixture</td>
<td>50</td>
<td>30% = 0.3</td>
<td>0.3(50) = 15</td>
</tr>
</tbody>
</table>
The chemist needs to obtain a 50-milliliter mixture.

The 50-milliliter mixture must be 30% sodium-iodine. The amount of sodium-iodine must be 30% of 50, or \((0.3)(50) = 15\) milliliters.

Step 3. Solve the system and answer the problem’s question. The system

\[
\begin{align*}
x + y &= 50 \\
0.1x + 0.6y &= 15
\end{align*}
\]

can be solved by substitution or addition. Let’s use substitution. The first equation can easily be solved for \(x\) or \(y\). Solving for \(y\), we obtain \(y = 50 - x\).

\[
y = \frac{50 - x}{0.1x + 0.6y} = 15
\]

We substitute \(50 - x\) for \(y\) in the second equation. This gives us an equation in one variable.

\[
0.1x + 0.6(50 - x) = 15
\]

This equation contains one variable, \(x\).

\[
0.1x + 30 - 0.6x = 15
\]

Apply the distributive property.

\[
-0.5x + 30 = 15
\]

Combine like terms.

\[
-0.5x = -15
\]

Subtract 30 from both sides.

\[
x = \frac{-15}{-0.5} = 30
\]

Divide both sides by \(-0.5\).

Back-substituting 30 for \(x\) in either of the system’s equations \((x + y = 50\) is easier to use) gives \(y = 20\). Because \(x\) represents the number of milliliters of the 10% solution and \(y\) the number of milliliters of the 60% solution, the chemist should mix 30 milliliters of the 10% solution with 20 milliliters of the 60% solution.

Step 4. Check the proposed solution in the original wording of the problem. The problem states that the chemist needs 50 milliliters of a 30% sodium-iodine solution. The amount of sodium-iodine in this mixture is \(0.3(50)\), or 15 milliliters. The amount of sodium-iodine in 30 milliliters of the 10% solution is \(0.1(30)\), or 3 milliliters. The amount of sodium-iodine in 20 milliliters of the 60% solution is \(0.6(20) = 12\) milliliters. The amount of sodium-iodine in the two solutions used in the mixture is 3 milliliters + 12 milliliters, or 15 milliliters, exactly as it should be.
STUDY TIP

Problems involving dual investments and problems involving mixtures are both based on the same idea: The total amount times the rate gives the amount.

**Dual Investment Problems:**

\[
\text{principal} \cdot \text{rate} = \text{interest}
\]

Percents are expressed as decimals in these equations.

**Mixture Problems:**

\[
\text{solution} \cdot \text{concentration} = \text{ingredient}
\]

Our dual investment problem involved mixing two investments. Our mixture problem involved mixing two liquids. The equations in these problems are obtained from similar conditions:

<table>
<thead>
<tr>
<th>Dual Investment Problems</th>
<th>Mixture Problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest from investment 1 + interest from investment 2 = amount of interest from mixed investments.</td>
<td>Ingredient amount in solution 1 + ingredient amount in solution 2 = amount of ingredient in mixture.</td>
</tr>
</tbody>
</table>

Being aware of the similarities between dual investment and mixture problems should make you a better problem solver in a variety of situations that involve mixtures.

CHECK POINT 3

A chemist needs to mix a 12% acid solution with a 20% acid solution to obtain 160 ounces of a 15% acid solution. How many ounces of each of the acid solutions must be used?

Problems Involving Motion

We have seen that the distance, \(d\), covered by any moving body is the product of its average rate, \(r\), and its time in motion, \(t\):

\[
d = rt \quad \text{Distance equals rate times time.}
\]

Wind and water current have the effect of increasing or decreasing a traveler’s rate.

EXAMPLE 4

Solving a Motion Problem

When a small airplane flies with the wind, it can travel 450 miles in 3 hours. When the same airplane flies in the opposite direction against the wind, it takes 5 hours to fly the same distance. Find the average rate of the plane in still air and the average rate of the wind.

**SOLUTION**

**Step 1. Use variables to represent unknown quantities.**

Let \(x\) = the average rate of the plane in still air.

Let \(y\) = the average rate of the wind.

**Step 2. Write a system of equations describing the problem’s conditions.**

A s it travels with the wind, the plane’s rate is increased. The net rate is its rate in still air, \(x\), plus the rate of the wind, \(y\), given by the expression \(x + y\). A s it travels against
the wind, the plane’s rate is decreased. The net rate is its rate in still air, \( x \), minus the rate of the wind, \( y \), given by the expression \( x - y \). Here is a chart that summarizes the problem’s information and includes the increased and decreased rates.

<table>
<thead>
<tr>
<th>Rate Description</th>
<th>Rate</th>
<th>Time</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trip with the Wind</td>
<td>( x + y )</td>
<td>3</td>
<td>( 3(x + y) )</td>
</tr>
<tr>
<td>Trip against the Wind</td>
<td>( x - y )</td>
<td>5</td>
<td>( 5(x - y) )</td>
</tr>
</tbody>
</table>

The problem states that the distance in each direction is 450 miles. We use this information to write our system of equations.

\[
3(x + y) = 450
\]

\[
5(x - y) = 450
\]

**Step 3. Solve the system and answer the problem’s question.** We can simplify the system by dividing both sides of the equations by 3 and 5, respectively.

\[
3(x + y) = 450 \quad \text{ Divide by 3. } \quad x + y = 150
\]

\[
5(x - y) = 450 \quad \text{ Divide by 5. } \quad x - y = 90
\]

Solve the system on the right by the addition method.

\[
\begin{align*}
3x & = 240 \\
x & = 120
\end{align*}
\]

Back-substituting 120 for \( x \) in either of the system’s equations gives \( y = 30 \). Because \( x = 120 \) and \( y = 30 \), the average rate of the plane in still air is 120 miles per hour and the average rate of the wind is 30 miles per hour.

**Step 4. Check the proposed solution in the original wording of the problem.** The problem states that the distance in each direction is 450 miles. The average rate of the plane with the wind is \( 120 + 30 = 150 \) miles per hour. In 3 hours, it travels \( 150 \cdot 3 \), or 450 miles, which checks with the stated condition. Furthermore, the average rate of the plane against the wind is \( 120 - 30 = 90 \) miles per hour. In 5 hours, it travels \( 90 \cdot 5 = 450 \) miles, which is the stated distance.

**Check Point 4** With the current, a motorboat can travel 84 miles in 2 hours. Against the current, the same trip takes 3 hours. Find the average rate of the boat in still water and the average rate of the current.

\[
\begin{align*}
\text{Distance} & = \text{Rate} \times \text{Time} \\
450 & = x \times 3 \\
150 & = x \\
\text{Distance} & = \text{Rate} \times \text{Time} \\
450 & = (x - y) \times 5 \\
90 & = x - y
\end{align*}
\]

\[
\begin{align*}
3x & = 240 \\
x & = 120 \\
y & = 30
\end{align*}
\]
Functions of Business: Break-Even Analysis  

As a young entrepreneur, did you ever try selling lemonade in your front yard? Suppose that you charged 55 cents for each cup and you sold 45 cups. Your revenue is your income from selling these 45 units, or $0.55(45) = $24.75. Your revenue function from selling x cups is

$$R(x) = 0.55x.$$ 

For any business, the revenue function, $R(x)$, is the money generated by selling x units of the product:

$$R(x) = px.$$ 

Back to selling lemonade and energizing the neighborhood with white sugar: Is your revenue for the afternoon also your profit? No. We need to consider the cost of the business. You estimate that the lemons, white sugar, and bottled water cost 5 cents per cup. Furthermore, mommy dearest is charging you a $10 rental fee for use of your (her?) front yard. Your cost function for selling x cups of lemonade is

$$C(x) = 10 + 0.05x.$$ 

For any business, the cost function, $C(x)$, is the cost of producing x units of the product:

$$C(x) = (\text{fixed cost}) + cx.$$ 

The term on the right, $cx$, represents variable cost, because it varies based on the number of units produced. Thus, the cost function is the sum of the fixed cost and the variable cost.

**REVENUE AND COST FUNCTIONS**  

A company produces and sells x units of a product.

**Revenue Function**

$$R(x) = (\text{price per unit sold})x$$

**Cost Function**

$$C(x) = \text{fixed cost} + (\text{cost per unit produced})x$$
Figure 3.7 shows the graphs of the revenue and cost functions for the lemonade business. Similar graphs and models apply no matter how small or large a business venture may be.

\[ R(x) = 0.55x \]
\[ C(x) = 10 + 0.05x \]

The lines intersect at the point (20, 11). This means that when 20 cups of lemonade are produced and sold, both cost and revenue are $11. In business, this point of intersection is called the **break-even point**. At the break-even point, the money coming in is equal to the money going out. Can you see what happens for \( x \)-values less than 20? The red cost graph is above the blue revenue graph. The cost is greater than the revenue and the business is losing money. Thus, if you sell fewer than 20 cups of lemonade, the result is a loss. By contrast, look at what happens for \( x \)-values greater than 20. The blue revenue graph is above the red cost graph. The revenue is greater than the cost and the business is making money. Thus, if you sell more than 20 cups of lemonade, the result is a gain.

Because the break-even point is the point of intersection of the graphs of revenue and cost functions, we can interpret each coordinate of this point. The \( x \)-coordinate of the point reveals the number of units that a company must produce and sell so that money coming in, the revenue, is equal to money going out, the cost. The \( y \)-coordinate of the break-even point gives the amount of money coming in and going out. Example 5 illustrates the use of the substitution method in determining a company’s break-even point.

**Example 5: Finding a Break-Even Point**

Technology is now promising to bring light, fast, and beautiful wheelchairs to millions of disabled people. A company is planning to manufacture these radically different wheelchairs. Fixed cost will be $500,000 and it will cost $400 to produce each wheelchair. Each wheelchair will be sold for $600.

**a.** Write the cost function, \( C \), of producing \( x \) wheelchairs.

**b.** Write the revenue function, \( R \), from the sale of \( x \) wheelchairs.

**c.** Determine the break-even point. Describe what this means.

**Solution**

**a.** The cost function is the sum of the fixed cost and variable cost.

\[ C(x) = 500,000 + 400x \]
b. The revenue function is the money generated from the sale of $x$ wheelchairs.

\[ R(x) = 600x \]

c. The break-even point occurs where the graphs of $C$ and $R$ intersect. Thus, we find this point by solving the system

\[
\begin{align*}
C(x) &= 500,000 + 400x \\
R(x) &= 600x \\
y &= 500,000 + 400x
\end{align*}
\]

Using substitution, we can substitute $600x$ for $y$ in the first equation.

\[
600x = 500,000 + 400x
\]

Substitute $600x$ for $y$ in $y = 500,000 + 400x$.

\[
200x = 500,000
\]

Subtract $400x$ from both sides.

\[
x = 2500
\]

Divide both sides by 200.

Back-substituting 2500 for $x$ in either of the system's equations (or functions), we obtain

\[
R(2500) = 600(2500) = 1,500,000.
\]

The break-even point is $(2500, 1,500,000)$. This means that the company will break even if it produces and sells 2500 wheelchairs. At this level, the money coming in is equal to the money going out: $1,500,000.

CHECK POINT 5 A company that manufactures running shoes has a fixed cost of $300,000. Additionally, it costs $30 to produce each pair of shoes. They are sold at $80 per pair.

a. Write the cost function, $C$, of producing $x$ pairs of running shoes.

b. Write the revenue function, $R$, from the sale of $x$ pairs of running shoes.

c. Determine the break-even point. Describe what this means.

What does every entrepreneur, from a kid selling lemonade to Donald Trump, want to do? Generate profit, of course. The profit made is the money taken in, or the revenue, minus the money spent, or the cost. This relationship between revenue and cost allows us to define the profit function, $P(x)$.

THE PROFIT FUNCTION The profit, $P(x)$, generated after producing and selling $x$ units of a product is given by the profit function

\[ P(x) = R(x) - C(x), \]

where $R$ and $C$ are the revenue and cost functions, respectively.
EXAMPLE 6 Writing a Profit Function

Use the revenue and cost functions for the lemonade business

\[ R(x) = 0.55x \] \quad \text{and} \quad \[ C(x) = 10 + 0.05x \]

to write the profit function for producing and selling x cups of lemonade.

SOLUTION The profit function is the difference between the revenue function and the cost function.

\[
P(x) = R(x) - C(x) \quad \text{This is the definition of the profit function.}
\]

\[
= 0.55x - (10 + 0.05x) \quad \text{Substitute the given functions.}
\]

\[
= 0.55x - 10 - 0.05x \quad \text{Distribute } -1 \text{ to each term in parentheses.}
\]

\[
= 0.50x - 10 \quad \text{Simplify.}
\]

The profit function is \( P(x) = 0.50x - 10 \).

The graph of the profit function, which can be expressed as \( y = 0.50x - 10 \), is shown in Figure 3.8. The red portion of the graph lies below the x-axis and shows a loss when fewer than 20 units are sold. The lemonade business is “in the red.” The black portion of the graph lies above the x-axis and shows a gain when more than 20 units are sold. The lemonade business is “in the black.”

CHECK POINT 6 Use the revenue and cost functions for the wheelchair business in Example 5

\[ R(x) = 600x \]

\[ C(x) = 500,000 + 400x \]

to write the profit function for producing and selling x wheelchairs.